Stability Results for Feedback Control Systems

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W. Griggs

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ABSTRACT

Several new stability results for system feedback interconnections are presented in this dissertation.

First, input-output stability results that capture a "blending" of the well-known small gain and passivity theorems, are provided. In the frequency domain, a system assumption of "small gain" over certain frequency intervals (as opposed to the entire frequency range) and "passivity" over the remaining frequency intervals (as opposed to the entire frequency range), is placed on each of two, stable, linear time-invariant (LTI) systems in a feedback interconnection. It is shown that input-output stability of the feedback interconnection follows. The frequency-dependent system assumption and associated input-output stability result are obtained by using a notion of dissipativity.

A "mixed" small gain and passivity assumption is then defined for causal, nonlinear systems in the time domain. An associated input-output feedback stability result is observed by placing a bound on the feedback system error and output signals in terms of bounded input signals.

The next main stability result concerns the standard stability robustness problem of subjecting an internally stable, nominal, LTI feedback control system to structured, linear time-varying (LTV) uncertainty. There exists (in the literature) a necessary and sufficient, scaled, small gain condition that determines robust stability of the nominal feedback-loop when subject to structured LTV perturbations. In this dissertation, the scaled small gain condition is used to formulate a (sufficient) stability robustness condition in a scaled LTI ν-gap metric framework. The scaled LTI ν-gap metric stability condition is shown to be computable via linear matrix inequality (LMI) techniques, similarly to the scaled small gain condition. Apart from a comparison with a generalized robust stability margin as the final part of the stability test, however, the solution algorithm implemented to test the scaled LTI ν-gap metric stability robustness condition is independent of knowledge about the controller transfer function (as opposed to the LMI feasibility problem associated with the scaled small gain condition which is dependent on knowledge about the controller). Thus, given a nominal plant and a structured uncertainty set, the stability robustness condition presented in this dissertation provides a single constraint on a controller (in terms of the generalized robust stability margin) such that *all* plants in the uncertainty set are (sufficiently guaranteed to be) stable.

Finally, in the case of single-input, single-output systems subject to output-multiplicative LTV uncertainty, the scaled LTI ν-gap metric condition is shown to be analytically computable.

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CONTENTS

NOTATION AND ACRONYMS

$\mathbb{R}$	real numbers
$\mathbb{R}_+$	non-negative real numbers
$\mathbb{C}$	complex numbers
$\mathbb{C}^{m\times n}$	matrix of complex numbers with m rows and n columns
j	the imaginary unit, ie: $j = \sqrt{-1}$
$Re(s)$	real part of $s \in \mathbb{C}$
$\bar{a}$	complex conjugate of a complex number a
$\exists$	there exists
$\forall$	for all
$:$	such that
$\in$	belongs to
$:=$	defined by
$\square$	end of proof
$<$	less than
$\leq$	less than or equal to
$>$	greater than
$\geq$	greater than or equal to
$\nless$	not less than
$\rightarrow$	maps to; tends to
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Leftrightarrow$	is equivalent to
$x \in (a,b)$	$a < x < b$ where $a, x, b \in \mathbb{R}$
$x \in (a,b]$	$a < x \leq b$ where $a, x, b \in \mathbb{R}$
$x \in [a,b)$	$a \leq x < b$ where $a, x, b \in \mathbb{R}$
$x \in [a,b]$	$a \leq x \leq b$ where $a, x, b \in \mathbb{R}$
$\min\{\ \}$	mimimum element of a set
$\max\{\ \}$	maximum element of a set
$\lim_{x\rightarrow a} f(x)$	limit of the function $f(x)$ as x tends to a
$\inf_{x\in\mathcal{X}} f(x)$	infimum of the function $f(x)$ over $x \in \mathcal{X}$
$\sup_{x\in\mathcal{X}} f(x)$	supremum of the function $f(x)$ over $x \in \mathcal{X}$
$\lvert\cdot\rvert$	absolute value; norm
0	zero matrix of compatible dimensions

$\mathbf{0}$	zero operator
I	identity matrix of compatible dimensions; identity operator
I_n	identity matrix of dimension $n \times n$
$[a_{ij}]$	ijth element of matrix A
A^T	transpose of matrix A
A^{-1}	inverse of matrix A
$\det(A)$	determinant of matrix A
$\mathrm{diag}(A_1, \ldots, A_n)$	block diagonal matrix or operator
$F_u(A, B)$	upper linear fractional transformation of matrices A and B
$F_l(A, B)$	lower linear fractional transformation of matrices A and B
$A \star B$	Redheffer star-product
$\mathcal{L}_p(\mathbb{R})$, $\mathcal{L}_p(\mathbb{R}_+)$	space of functions $\{f : \mathbb{R} \to \mathbb{R}\}$ such that $t \to \lvert f(t)\rvert^p$ is integrable over $\mathbb{R}$, $\mathbb{R}_+$ respectively; typically $p = 1, 2, \infty$
$\mathcal{L}_2(j\mathbb{R})$	frequency domain Hilbert space of square integrable functions on $j\mathbb{R}$ including ∞
$\mathcal{H}_2$	subspace of $\mathcal{L}_2(j\mathbb{R})$ with functions analytic in $Re(s) > 0$ (ie: open right-half plane)
$\mathcal{L}_\infty(j\mathbb{R})$	frequency domain Banach space of functions that are (essentially) bounded on $j\mathbb{R}$ including ∞
$\mathcal{H}_\infty$	subspace of $\mathcal{L}_\infty(j\mathbb{R})$ with functions that are analytic and bounded in $Re(s) > 0$
$\mathcal{R}$	set of proper real rational transfer function matrices
$\mathcal{R}^{m \times n}$	set of proper real rational transfer function matrices with m rows and n columns
prefix $\mathcal{R}$	subspace of real rational functions, eg: $\mathcal{RH}_\infty$
$\times$	product domain
IQC	integral quadratic constraint
LFT	linear fractional transformation
LHS	left-hand side
LMI	linear matrix inequality
LTI	linear time-invariant
LTV	linear time-varying
MIMO	multi-input, multi-output
RHP	right-half plane
RHS	right-hand side
SISO	single-input, single-output

1. INTRODUCTION

1.1 Background

Stability results for feedback control systems are provided in this dissertation. Feedback is a basic concept of control theory. Output variables of a system to be controlled are measured, and this information is processed to generate an input to the system to be controlled such that the overall set-up behaves in some desired fashion. A model applicable to most feedback systems is shown in Fig. 1.1. Here, u_1 and u_2 denote (external) input signals, e_1 and e_2 denote error signals, and y_1 and y_2 are output signals. The notation K is associated with the controller and the notation P is associated with the system to be controlled. The feedback control system pictured in Fig. 1.1 is referred to as a negative feedback control system.

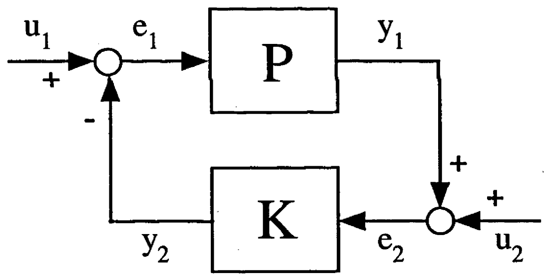

Fig. 1.1: Feedback control system.

Feedback is used for a number of reasons. One of these is to reduce the effect of any unmeasured disturbances acting on the system [86]. Another is to reduce the effect of any uncertainty about the system dynamics [86]. That is, an aim of using feedback is to minimize the effects of lack of knowledge about a system which is to be controlled. In the absence of uncertainty then, there may be no need for feedback, and decisions could be made "open-loop" [95]. However, systems (plants, sensors or actuators) are not free of uncertainty [95].

Two basic properties of feedback systems are well-posedness and stability. Feedback system well-posedness is essential as it relates to whether a mathematical model is adequate as a description of a physical system. More accurately, it corresponds to a question of existence and uniqueness of solutions to the equations

$$e_1 = u_1 - y_2 \tag{1.1}$$
$$e_2 = u_2 + y_1 \tag{1.2}$$

which describe the feedback system shown in Fig. 1.1, together with the equations

$$y_1 = Pe_1$$
$$y_2 = Ke_2.$$

(That is, it corresponds to a question of existence and uniqueness of solutions for e_1, e_2, y_1, y_2 for each choice of u_1, u_2) [6, 39, 84, 85, 95]. It is usual in the definition of well-posedness to further require that the errors and the outputs depend on the inputs in a non-anticipatory way; and that the errors and the outputs depend, on finite intervals, Lipschitz continuously on the inputs. (Lipschitz continuity is a smoothing condition for functions that is stronger than regular continuity. Intuitively, a Lipschitz continuous function is limited in how fast it can change: a line joining any two points on the graph of this function will never have a slope steeper than a certain number called a Lipschitz constant of the function.) References [6, 39, 84, 85, 95] provide conditions to impose on P and K to guarantee well-posedness of the feedback-loop. Well-posedness is not discussed in detail in this dissertation; it is assumed of most feedback interconnections under consideration.

Stability is a desired property of a feedback control system. The study of system stability has a rich history and there are many different notions of stability of systems. In all cases, the idea of determining stability involves determining whether a system is well-behaved in some sense, given a set of system equations.

The different notions of stability are often based on the way a physical system is mathematically described. Two important ways of mathematically describing physical systems are as follows. The first way is to give an internal description of the physical system. This approach uses the physical laws and internal interconnections governing the system as the basis of the mathematical model. Accordingly, this description generally takes the form of an ordinary differential equation or a partial differential equation. Also, one works with a set of intermediate variables (related to the concept of state). As a result, there are two parts to mathematical models that internally describe systems: a dynamical part, which describes the evolution of the state under the influence of the inputs; and a memoryless part, which relates the output to the state (and sometimes to the instantaneous value of the input as well).

Stability, in relation to the internal (or state-space) description of a system, is regarded as an internal property. The system is considered as excited by an initial condition, and boundedness or convergence of the state for future time is taken as the basic stability requirement. In other words, stability in the state-space description sense is concerned with the behavior of trajectories of a system when its initial state is near equilibrium; the object is thus to draw conclusions about the behavior of a system without actually focusing on particular solution trajectories. From a practical viewpoint, stability in this sense is important because external disturbances such as noise, wind and component errors are always present in a real system to perturb it from equilibrium.

One of the founders of stability theory in the internal description sense was the Russian mathematician A. M. Lyapunov [62]. He introduced many of the basic definitions of stability that are in use today, and also proved many of the fundamental theorems. Some extraordinary contributions to the field were made by V. A. Yakubovich, V. M. Popov and R. E. Kalman (see [55, 74, 97] for example). Modern systems theory relies heavily on the state formulation for synthesis techniques, as illustrated by some of the highlights of modern control theory: for example, Pontryagin's maximum principle [73]; the regulator problem for linear systems [57]; and the Kalman-Bucy filtering theory [56, 58].

The input-output approach is the second way to mathematically describe a physical system. Here, the mathematical model usually takes the form of an operator equation expressing the relationship between the inputs (the variables to be manipulated) and the outputs (the variables of interest). Such a description is often obtained from some representative experiments. Significantly, the input-output approach relates external variables: the system is viewed as a "black box" and the description does not depend in any way on the notion of state. In other words, this approach requires minimal knowledge of the physical laws governing the system and of the interconnections within the "black box". The input-output description provides the benefits of abstraction: because it is free of details about the internal description, basic results in system theory can be viewed more easily. In system design, this approach facilitates designing for a prescribed response to a specified class of inputs.

Stability, in relation to the input-output description of systems, is regarded as an input-output property. As the system is naturally regarded as a mapping between normed spaces, the boundedness of this map is taken as the basic requirement for stability. (The boundedness of the mapping then yields a bound on the norm of the output in terms of the norm of the input.) The successful development of input-output stability theory occurred much more recently than the development of Lyapunov theory. It was pioneered by I. W. Sandberg [79] and G. Zames [100, 101] in the 1960's.

Relationships between the internal and input-output notions of stability exist: J. C. Willems, for example, illuminated some of these relationships in [91] (the input-output and state space representations of systems are not interchangeable, however [99]). For instance, the circle criterion, Popov criterion, passivity theorem and small gain theorem are a few of the important stability results obtained over the last few decades. These results have counterparts in each of the internal and the input-output approaches (although the abstract results on passivity and system gain emerged via input-output methods; for example, see [102, 103]).

1.2 Aim and Motivations

The aim of this dissertation is to propose several stability results for feedback system interconnections, and in so doing, expand on the feedback control system stability theory available. Two distinct stability problems are addressed. The first is posed and solved based on the input-output systems theory approach. Internal, or state-space, system descriptions are considered in the second problem. A brief description of each of the two problems is provided below, with motivation for their study, and summaries of some of the key techniques used in this dissertation to solve them. A review of some of the key methods used in the literature to solve similar problems is also provided where appropriate.

1.2.1 Problem 1: "Mixed" small gain and passivity properties

The first problem involves determining input-output stability of a negative feedback interconnection as shown in Fig. 1.1 (or as shown below in Fig. 1.2). The abstract reasoning of the input-output approach has lead to the development of some very important stability theorems such as the small gain theorem [100, 101] and the passivity theorem [15, 78, 100, 101]. It is possible to set up a general input-output framework which supports results such as these. Consider the negative feedback interconnection shown in Fig. 1.2. The input, error and output signals are functions of time, defined for $t \geq 0$; and they take values in some (real-valued) normed space. Recall that two of the four equations describing the feedback system shown in Fig. 1.2 were given by (1.1) and (1.2). The symbols M_1 and M_2 are operators acting on their respective inputs e_1 and e_2 to produce outputs y_1 and y_2, respectively. Then the input-output stability problem is to show that if u_1, u_2 belong to some class of functions (such as the $\mathcal{L}_p$ spaces), then e_1, e_2 and y_1, y_2 also belong to the same class of functions.

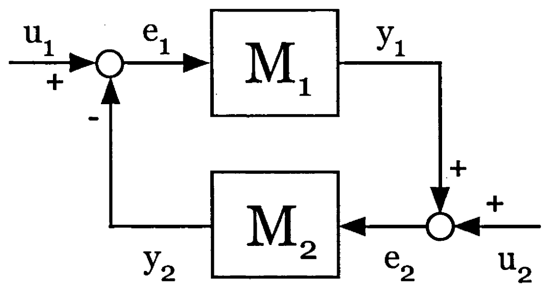

Fig. 1.2: Negative feedback interconnection.

Typically, input-output stability results are obtained by assuming that systems have specific properties associated with them. For instance, the small gain theorem ensures stability provided that the product of the gains of the two systems in the feedback interconnection is less than one. The passivity theorem guarantees stability, for example, if both systems in the feedback interconnection are passive, and one of them is input strictly passive with finite gain. Of course, there exist many situations where stability cannot be determined by use of the small gain or passivity theorems because the assumptions required on systems as stated in the theorems do not appropriately match the properties of the actual systems in the feedback interconnection in question. In this dissertation, small gain and passivity properties are "blended" in an appropriate way as to create a (super) class of system assumptions (which captures systems described by small gain concepts and passivity concepts). Input-output stability results of feedback interconnections are consequently derived.

Obtaining such results has practical applicability. For instance, it has been observed that high frequency dynamics can frequently destroy the passivity property of an otherwise passive system. A celebrated controversy in adaptive control [77] depended on the observation that passivity conditions, normally forming part of the hypotheses used in the proofs of convergence of certain adaptive control algorithms, should not be assumed to be valid in practice (because high frequency dynamics often neglected for modelling purposes will always be present in a real system). Failure of the passivity condition invalidated the applicability of the associated theorem on the algorithm convergence to most real-life applications, and left a cloud hanging over the real-life use of the algorithm. Simulations of [77] confirmed that adverse behavior could occur when high frequency dynamics were explicitly taken into account.

The book [4] (see also [63] and [1]) described tools for establishing stability of adaptive systems of the type examined in [77]; that is, where "passivity" properties hold only for low frequency signals. Stability is established if additionally (and in a rough manner of speaking), "gains" are small at high frequencies (ie: a small gain property in the sense of the small gain theorem holds in the frequency band where the passivity condition fails). Thus, an important class of applications in which passivity and small gain ideas have to be "blended" has been illustrated.

Extensions to stability-associated results to accommodate system properties in specific frequency bands have recently been developed. For example, [51] extended the "static" concept of dissipativity to a "dynamic", frequency-dependent framework for linear time-invariant (LTI) systems.

Dissipativeness is a property, presented in this thesis as an input-output property of a general dynamical system, which captures concepts such as passivity and finite gain. The study of dissipative systems was initiated by J. C. Willems [93] in order to tie together ideas common to network theory and feedback control theory, as well as thermodynamics and mechanics. (Beyond the dissipative systems theory associated with network synthesis of the 1930s, this work can be seen as evolving from a series of studies, beginning with the Kalman-Yakubovich lemma [55, 96] and its applications [5, 7, 22, 94], which can be interpreted as exploring the usefulness of the concept of passivity or positive real transfer functions.) In [93], dissipativeness was defined as essentially a generalization of the property of passivity via an inequality based on a state-space description. In other words, dissipativeness was introduced as a concept which reflected something of the internal properties of the system.

In [43–45, 68, 70], dissipative systems theory was utilized to produce general stability results for interconnected systems. In the process, extensions to the theory in [92, 93] were made. These extensions included the consideration of dissipative systems in a purely operator theoretic setting; clarification of the role of minimality of the state-space representation; and the providing of algebraic tests for dissipativeness of classes of nonlinear systems. (In addition to the papers on interconnected systems stability, some of these extensions are found in [46].) Much of the work involved carrying over known results for special cases to the more general situation to provide as general a framework as possible for applications. Extensions to the theory in [92, 93] were also provided in [91]. The purpose of [42] was to collect together and further extend the essential features of the theory of dissipative systems. (Some of the results were only variations of those given by [91–93], but the overall intention was to provide a complete background for applications of the theory.) In [40], new results on the instability of general interconnected systems, derived in terms of dissipative systems theory, were presented.

The paper [47] provides another example of where extensions to stability associated results to accommodate for system properties in specific frequency bands have been developed. The paper [47] generalized the Kalman-Yakubovich-Popov lemma to establish a relationship between a frequency domain inequality in a finite frequency range, and a linear matrix inequality (LMI) condition. (The standard Kalman-Yakubovich-Popov lemma treats frequency domain inequalities, which characterize various properties of dynamical systems, for the entire frequency range only.) See also [48–50, 81, 98] for results regarding restricted frequency ranges.

On another note, the use of integral quadratic constraints (IQCs) to describe systems in feedback interconnections was introduced in [67] as a powerful method of determining closed-loop stability. The result assumes that one of the systems in the feedback interconnection is described by a LTI operator, while the other system represents the "trouble-making" (nonlinear, time-varying or uncertain) components of the feedback loop. The stability theorem [67, Theorem 1] then captures the classical small gain and passivity/dissipativity theorems under the proviso that one of the two cascaded systems in the loop is LTI.

The "blended" small gain and passivity properties described in this thesis are referred to as "mixed" small gain and passivity properties. In the first instance (motivated by a desire to accommodate for frequency range specific systems properties), a LTI "mixed" small gain and passivity frequency domain property is defined using the notion of dissipativity. It is shown that (finite-gain, and hence) input-output stability of a feedback interconnection consisting of two multi-input, multi-output (MIMO) LTI systems with "mixed" small gain and passivity frequency domain properties is guaranteed. The interconnected dissipative systems approach (as opposed to an IQC approach, which would seem readily possible) is used, as the methodology paves the way for a similar result when the systems are nonlinear. Consequently, a "mixed" small gain and passivity time domain property is defined for nonlinear systems, and it is shown that input-output stability of a feedback interconnection consisting of two nonlinear systems with these "mixed" properties is certain.

1.2.2 Problem 2: Linear time-varying uncertain systems

When a stabilizing controller is designed for a nominal plant, a desired objective is that the controller also succeeds in stabilizing the "true-life" system in the face of uncertainty. Uncertainty may be modelled as an unstructured perturbation to the nominal plant; classes of these uncertainties include additive uncertainty, input- or output-multiplicative uncertainty, and input- or output-feedback uncertainty. A structured uncertainty model may be used when plants are subjected to multiple uncertainties, for example when the plant contains multiple unstructured uncertainties, or when the plant contains a number of uncertain parameters.

The second problem addressed in this dissertation involves determining internal stability of a system subject to bounded linear time-varying (LTV) uncertainties, given that a nominal feedback interconnection, consisting of a LTI system P and a LTI controller K as shown in Fig. 1.1, is internally stable. Often it is suitable to describe such a problem using a linear fractional transformation (LFT) framework, as shown in Fig. 1.3, where $F(s)$ is a transfer function matrix that describes the relationship between the nominal LTI plant P and the structured LTV uncertainty denoted by Δ.

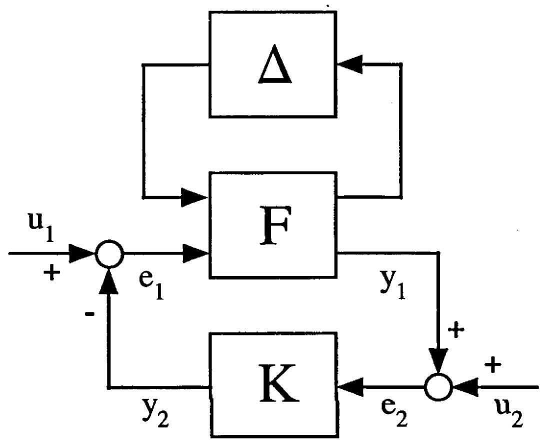

Fig. 1.3: Uncertain system.

This type of stability problem has been studied intensively in the literature. For instance, [8] reduced the problem (where Δ was possibly nonlinear) to a question of existence of a quadratic Lyapunov function of a certain structure. The existence of the Lyapunov function was determined by solving a convex optimization problem. In [71], where complex-valued uncertainty was considered, quadratic stability (which is related to the existence of quadratic Lyapunov functions) was shown to be equivalent to a scaled H_∞ norm condition when the structured uncertainty consisted of only two diagonal blocks. (The equivalence for more than two blocks is not in general true [71].)

A frequency domain stability criterion based on IQCs was derived in [54], where an uncertainty structure consisting of bounded, real-valued, differentiable, time-varying parameters (with bounded derivatives) down the diagonal was considered.

The paper [66] considered structured slowly time-varying uncertain gains and obtained a sufficient frequency domain condition for stability when pairs of the uncertain gain and its derivative belonged to a given convex set. A sufficient stability condition (that could be formulated in terms of LMIs) was derived in [9], and was shown to be less conservative than a standard scaled small gain stability condition when the uncertainty structure contained real, repeated, time-varying parameters (ie: when sub-blocks of the uncertainty structure shared the same scalars). Obtaining this condition did not require the use of IQCs [54], or the explicit construction of a quadratic Lyapunov function [71]; but followed from basic properties of the structured singular value (although the results are closely related to notions of quadratic stability - [71] used the quadratic stability approach to derive the condition for the case where all of the parameters are complex).

In [83], a computational approach was developed for designing a globally optimal controller that was robust to time-varying nonlinear perturbations in the plant. The controller design problem was formulated as an optimization with bilinear matrix inequality constraints, and solved to optimality by a branch-and-bound algorithm (see [24] for instance). A branch-and-bound scheme was also used in [59] to obtain a globally optimal solution to a robust synthesis problem.

The main contribution of this thesis in regards to the stability robustness problem shown in Fig. 1.3 (where Δ is structured LTV uncertainty, and F and K are LTI) concerns the development of a sufficient scaled LTI ν-gap metric stability robustness condition. This scaled LTI ν-gap metric condition is an extension of a standard, necessary and sufficient, scaled small gain condition (which is described in [17,20,80]). An advantage of the scaled LTI ν-gap metric condition is that, apart from a comparison with a generalized robust stability margin as the final part of the stability test, the solution algorithm implemented to test the condition is independent of the controller.

The LTI ν-gap metric is introduced in [90]. Like its predecessor the gap metric [21, 26, 28, 31, 105], the ν-gap metric offers a measure of difference or "distance" between two systems from a feedback perspective, and thus provides a means of quantifying feedback system stability robustness. Any plant at a ν-gap distance less than, say β, from the nominal plant will be stabilized by any controller which stabilizes the nominal with a stability margin of β. Unlike the gap metric, it can also be said of the ν-gap metric that any plant at a distance greater than β from the nominal will be destabilized by some controller which stabilizes the nominal with a stability margin of at least β [90]. In this sense, the LTI ν-gap metric is less conservative than the gap metric. The LTI ν-gap metric is also simpler to compute. Time-varying and nonlinear extensions to both the gap metric [23,25,27,29,52] and the ν-gap metric [3, 87, 88] exist. Analytical computations of the metrics in these cases are generally not possible.

1.3 Organization of the Dissertation

The dissertation is organized as follows. Chapter 2 contains important stability results and associated mathematical preliminaries from the literature, relevant to this dissertation. The content of Chapters 3 and 4 corresponds to Problem 1, described above. A "mixed" small gain and passivity frequency domain property for LTI systems, and the associated input-output stability result, is presented in Chapter 3. In Chapter 4, a "mixed" small gain and passivity time domain property for nonlinear systems, and the associated stability result, is provided. The content of Chapters 5 and 6 is associated with Problem 2. In Chapter 5, a standard scaled small gain stability robustness condition is extended into a LTI ν-gap metric framework: a scaled LTI ν-gap metric condition is provided that determines stability of a feedback interconnection subject to LTV uncertainty. It is shown that this condition can be checked by solving a LMI feasibility problem. In Chapter 6, the part of the scaled LTI ν-gap metric condition (presented in Chapter 5) that requires solution via solving a LMI feasibility problem, is shown to be analytically computable when all plants considered are single-input, single-output (SISO), and the LTV uncertainties are of the output-multiplicative type. Chapter 7 summarizes the contributions of the dissertation and outlines potential directions for future research. The appendix contains calculations that are relevant to include, but disruptive to the main flow of arguments in the dissertation.

2. BASIC STABILITY RESULTS

This chapter presents important stability results and associated mathematical preliminaries from the literature that are relevant to this thesis.

2.1 *Input-output Stability Results*

A desired property of a feedback interconnection of two systems is that the interconnection is input-output stable [95]. To determine stability, one typically places assumptions on the two systems in the interconnection; and shows that, if the closed-loop system's inputs belong to some class of functions (such as the $\mathcal{L}_p$ spaces), then the errors and outputs also belong to the same class of functions [19]. To illustrate, a negative feedback interconnection is shown in Fig. 4.1, where H_1 and H_2 are operators acting on the errors e_1 and e_2, respectively, to produce outputs y_1 and y_2, respectively.

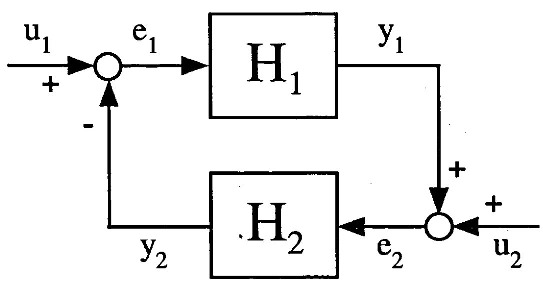

Fig. 2.1: Input-output stability framework.

The small gain and passivity theorems are two stability results identifiable by the assumptions placed on the systems in the feedback interconnection. Below, the small gain and passivity theorems are stated. In the form provided, we see that they give sufficient conditions under which a system "bounded input" produces a "bounded output". The formulations are chosen so that the question of boundedness is completely disconnected from the questions of existence, uniqueness, etc. [19]. (This also applies to the formulation of stability theorems associated with system "mixed" small gain and passivity properties, discussed later in this thesis.)

We select our working signal space to be the $\mathcal{L}_2$-space in particular (although frequently the $\mathcal{L}_p$-spaces in general are considered). Recall that $\mathcal{L}_2[0,\infty)$ is a Lebesgue space with inner product defined as

$$\langle f, g\rangle = \int_0^\infty g'(t)f(t)dt,$$

where the superscript $(\cdot)'$ denotes the vector transpose. The norm of functions in $\mathcal{L}_2[0,\infty)$ is denoted by $\|\cdot\|$, where $\|f\|^2 := \langle f, f\rangle$. For $T \in [0,\infty)$, P_T denotes the truncation operator. That is, for a function $f(t)$, $0 \leq t < \infty$,

$$(P_T f)(t) := \begin{cases} f(t), & t \leq T \\ 0, & t > T \end{cases}.$$

For convenience, the notation $f_T := P_T f$ is used. Then $\mathcal{L}_{2e}$ denotes the extension of the space $\mathcal{L}_2[0,\infty)$, defined by $\mathcal{L}_{2e} := \{f : f_T \in \mathcal{L}_2[0,\infty)\ \forall T \in [0,\infty)\}$.

Definition 1. *[64, Definition 6.5] A system, or more precisely, the mathematical representation of a physical system, is defined to be a mapping $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, that satisfies the so-called causality condition*

$$(Hf(\cdot))_T = (Hf_T(\cdot))_T$$

for all $f \in \mathcal{L}_{2e}$ and all $T \in [0,\infty)$.

Definition 2. *[64, Definition 6.6] A system $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to be input-output $\mathcal{L}_2$-stable if, whenever the input belongs to $\mathcal{L}_2[0,\infty)$, the output is once again in $\mathcal{L}_2[0,\infty)$ (ie: H is input-output $\mathcal{L}_2$-stable if $Hf \in \mathcal{L}_2[0,\infty)$ whenever $f \in \mathcal{L}_2[0,\infty)$).*

For simplicity, input-output $\mathcal{L}_2$-stability will be referred to as input-output stability, or simply stability, when the context is clear.

Theorem 1. (Small Gain Theorem) *[19] Consider the feedback interconnection shown in Fig. 2.1. Let $H_1, H_2 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$. Let $e_1, e_2 \in \mathcal{L}_{2e}$ and define u_1 and u_2 by*

$$\begin{aligned} u_1 &= e_1 + H_2 e_2 \\ u_2 &= e_2 - H_1 e_1. \end{aligned}$$

Suppose that there are constants $\eta_1, \eta_2, \epsilon_1 \geq 0, \epsilon_2 \geq 0$ such that

$$\begin{aligned} \|(H_1 e_1)_T\| &\leq \epsilon_1 \|e_{1T}\| + \eta_1 \\ \|(H_2 e_2)_T\| &\leq \epsilon_2 \|e_{2T}\| + \eta_2 \end{aligned}$$

$\forall T \in [0, \infty)$. *Under these conditions, if* $\epsilon_1\epsilon_2 < 1$, *then*

$$\|e_{1T}\| \leq (1 - \epsilon_1\epsilon_2)^{-1}(\|u_{1T}\| + \epsilon_2\|u_{2T}\| + \eta_2 + \epsilon_2\eta_1) \tag{2.1}$$

$$\|e_{2T}\| \leq (1 - \epsilon_1\epsilon_2)^{-1}(\|u_{2T}\| + \epsilon_1\|u_{1T}\| + \eta_1 + \epsilon_1\eta_2) \tag{2.2}$$

$\forall T \in [0, \infty)$. *Furthermore, if* $\|u_1\|, \|u_2\| < \infty$ *then* e_1, e_2 *and* y_1, y_2 *have finite norms, and the norms of the errors are bounded by the right hand sides of* (2.1) *and* (2.2), *provided all subscripts* T *are dropped.*

Proof of the small gain theorem is provided in [19]. Before proceeding with a statement of the passivity theorem, we define $\langle f, g\rangle_T := \langle f_T, g_T\rangle$ and note that $\langle f_T, g_T\rangle = \langle f_T, g\rangle = \langle f, g_T\rangle$.

Theorem 2. (Passivity Theorem) *[19] Consider a feedback system as shown in Fig. 2.1 and described by*

$$e_1 = u_1 - H_2 e_2$$
$$e_2 = u_2 + H_1 e_1$$

where $H_1, H_2 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$. *Assume that for any* $u_1, u_2 \in \mathcal{L}_2[0, \infty)$, *there are solutions* $e_1, e_2 \in \mathcal{L}_{2e}$. *Suppose that there are constants* $\epsilon_1, \eta_1, l_1, \bar{\eta}_1, k_2, \bar{\eta}_2$ *such that*

$$\|(H_1 f)_T\| \leq \epsilon_1\|f_T\| + \eta_1$$
$$\langle f, H_1 f\rangle_T \geq l_1\|f_T\|^2 + \bar{\eta}_1$$
$$\langle H_2 f, f\rangle_T \geq k_2\|(H_2 f)_T\|^2 + \bar{\eta}_2$$

$\forall f \in \mathcal{L}_{2e}$, $\forall T \in [0, \infty)$. *Under these conditions, if*

$$l_1 + k_2 > 0 \tag{2.3}$$

then $u_1, u_2 \in \mathcal{L}_2[0, \infty)$ *imply that* $e_1, e_2, H_1 e_1, H_2 e_2 \in \mathcal{L}_2[0, \infty)$.

For a proof, see [19].

Remark 1. *When* $k_2 = 0$, (2.3) *requires that* $l_1 > 0$; *then the theorem holds if* H_1 *is input strictly passive with finite gain and* H_2 *is passive.*

The notions of passivity, input strict passivity and finite gain are formally defined in Chapter 4.

2.2 The LTI ν-gap Metric and Associated Stability Results

The LTI ν-gap metric introduced by [90] provides a means of quantifying feedback system stability robustness in terms of offering the control system engineer a measure of difference or "distance" between two systems from a feedback perspective, as follows.

2.2.1 Generalized Robust Stability Margin

Let a feedback interconnection consisting of a nominal LTI plant P_0 and a LTI controller K, as shown in Fig. 2.2, be denoted by $[P_0, K]$. This interconnection is said to be internally stable if it is well-posed and each of the four transfer functions mapping the signals v_1 and v_2 to y and u are stable; that is, they belong to $\mathcal{RH}_\infty$ [104, Lemma 5.3]. The generalized robust stability margin $b_{P_0,K}$ [30, 75, 86] is given by

$$b_{P_0,K} := \left\| \begin{pmatrix} I \\ P_0 \end{pmatrix} (I - KP_0)^{-1} \begin{pmatrix} I & K \end{pmatrix} \right\|_\infty^{-1}$$

if $[P_0, K]$ is internally stable; and by $b_{P_0,K} := 0$ otherwise. It is also possible to define $b_{opt}(P_0) := \sup_K b_{P_0,K}$ [86]. It is shown in [30] that $b_{opt}(P_0) < 1$ for any P_0.

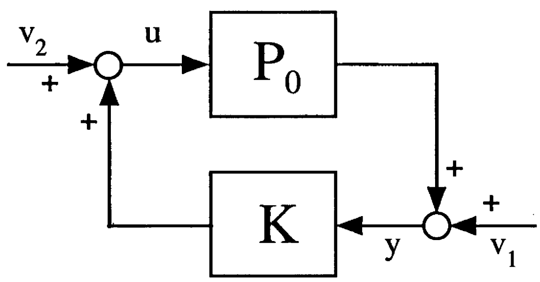

Fig. 2.2: Internal stability of $[P_0, K]$.

2.2.2 The LTI ν-gap Metric

A convenient formulation of the ν-gap metric, $\delta_\nu(P_0, P_1)$, between two systems $P_0, P_1 \in \mathcal{R}^{n\times m}$, is given by: $\delta_\nu(P_0, P_1) := \|\tilde{G}_1 G_0\|_\infty$ if $\det(G_1^* G_0)(j\omega) \neq 0 \ \forall \omega \in (-\infty, \infty)$ and $\mathrm{wno}(\det(G_1^* G_0)) = 0$; and by $\delta_\nu(P_0, P_1) := 1$ otherwise [86, 90]. Here $G_i, \tilde{G}_i$ denote normalized right and left graph symbols, respectively, for plants P_i, $i = 0, 1$ (where an account of normalized right and left graph symbols is given in [86]); and $\mathrm{wno}(g)$ denotes the winding number about the origin of $g(s)$ (or number of encirclements of the origin made by $g(s)$), as s follows the standard Nyquist D-contour (see [86, Section 1.2.2] for more details). An efficient state-space method for computing $\delta_\nu(P_0, P_1)$ is provided in [86, Appendix A.2].

2.2.3 Stability Results

The generalized robust stability margin, $b_{P_0,K}$, and the ν-gap metric, $\delta_\nu(P_0, P_1)$, are related to each other by the inequality

$$b_{P_1,K} \geq b_{P_0,K} - \delta_\nu(P_0, P_1); \tag{2.4}$$

and by the stronger inequality

$$\arcsin b_{P_1,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P_1)$$

(the second inequality implies the first) [86, 90]. Inequality (2.4) demonstrates that a feedback interconnection $[P_1, K]$ is internally stable provided (the feedback interconnection $[P_0, K]$ is internally stable, and) $\delta_\nu(P_0, P_1)$ is strictly less than $b_{P_0,K}$. In fact, for a given controller K that achieves $b_{P_0,K}$ (where $b_{P_0,K} > \beta \geq 0$), the set $\{P : \delta_\nu(P_0, P) \leq \beta\}$ is a neighborhood or "ball" of systems about P_0 that are guaranteed to achieve a generalized robust stability margin of at least $b_{P_0,K} - \beta$ with K. These points correlate with part (i) of the following lemma.

Lemma 1. *[86, Remark 3.11]*

(i) Given a nominal plant $P_0 \in \mathcal{R}^{n\times m}$, a controller $K \in \mathcal{R}^{m\times n}$ and some number $\beta \in (0, b_{opt}(P_0))$, then $[P_1, K]$ is internally stable for all plants $P_1 \in \mathcal{R}^{n\times m}$ satisfying $\delta_\nu(P_0, P_1) \leq \beta$ if and only if $b_{P_0,K} > \beta$.

(ii) Given a nominal plant $P_0 \in \mathcal{R}^{n\times m}$, a perturbed plant $P_1 \in \mathcal{R}^{n\times m}$ and a positive number $\beta < b_{opt}(P_0)$, then $[P_1, K]$ is internally stable for all controllers $K \in \mathcal{R}^{m\times n}$ satisfying $b_{P_0,K} > \beta$ if and only if $\delta_\nu(P_0, P_1) \leq \beta$.

Part (ii) of the above lemma states that any plant at a distance greater than β from the nominal system (as measured by the ν-gap metric), is guaranteed to be destabilized by at least one controller which stabilizes the nominal system with a stability margin of at least β. The gap metric shares property (i) with the ν-gap metric; while property (ii) is unique to the ν-gap metric alone [86].

3. A "MIXED" SMALL GAIN AND PASSIVITY FREQUENCY DOMAIN PROPERTY FOR LTI SYSTEMS[1]

3.1 Introduction

Two of the most important results in the input-output stability theory of interconnected systems are the small gain and passivity theorems. The small gain theorem states that if the product of the gains of two stable systems is less than one then the feedback interconnection of the two systems is stable [19, 32, 64, 104] (see Theorem 1). The passivity theorem guarantees stability of a feedback interconnection of two stable systems if, for instance, both of the systems are passive, and one of them is input strictly passive with finite gain [19, 32, 64, 82] (see Theorem 2). Of course, there exist many situations where stability of an interconnection cannot be guaranteed by use of the small gain or passivity theorems alone because the properties of the systems in the feedback-loop in question are not compatible. One instance is the adaptive control example given in Section 1.2.1.

The idea of merging the passivity and small gain theorems to provide stability results for feedback interconnections containing systems belonging to a class that encompasses those dealt with by the small gain theorem and passivity theorems alone, would therefore be extremely useful. For example, consider two open-loop, causal, stable, single-input single-output (SISO) systems with LTI transfer functions, say

$$m_1(s) = \frac{3}{(s+1)(s+2)}$$

and

$$m_2(s) = \frac{13}{(s+3)(s+4)}$$

with Nyquist diagrams shown in Figs. 3.1 and 3.2. It is clear that, if in some frequency range $[0, \Omega]$ the systems are passive (ie: the real part of each of the transfer functions is positive), and if in the frequency range $[\Omega, \infty)$ the product of the amplitudes of the transfer functions is less than one, then there is no way that

[1] Parts of this chapter are to be published in the proceedings of [34]; and parts have been accepted for publication in *Systems & Control Letters* [35].

the Nyquist diagram of the cascade would encircle the point $-1 + j0$. Accordingly, the closed-loop would be stable. (Indeed, as depicted in Figs. 3.1 and 3.2, such an Ω exists, and the Nyquist diagram of $m_1(s)m_2(s)$ shown in Fig. 3.3 does not encircle $-1 + j0$.)

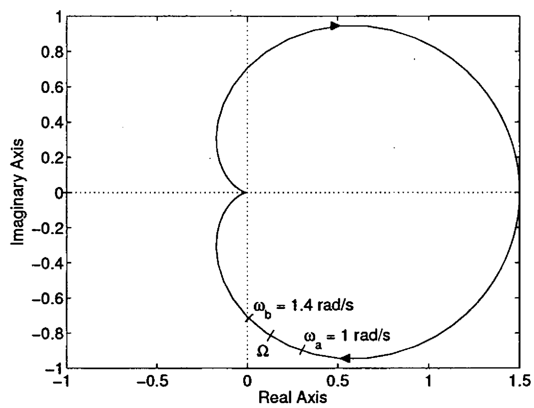

Fig. 3.1: Nyquist diagram of $m_1(s)$.

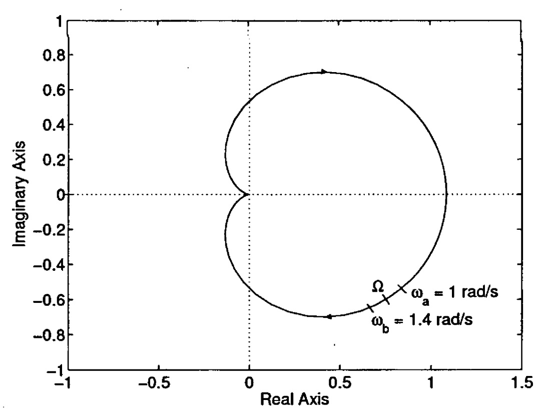

Fig. 3.2: Nyquist diagram of $m_2(s)$.

Obviously, however, $m_1(s)$ and $m_2(s)$ are not of a form that allows treatment of closed-loop stability by the small gain or passivity theorems. Furthermore, one could not simply scale one of the systems with transfer functions $m_1(s)$ or $m_2(s)$ to have gain less than one in order to determine closed-loop stability. This would

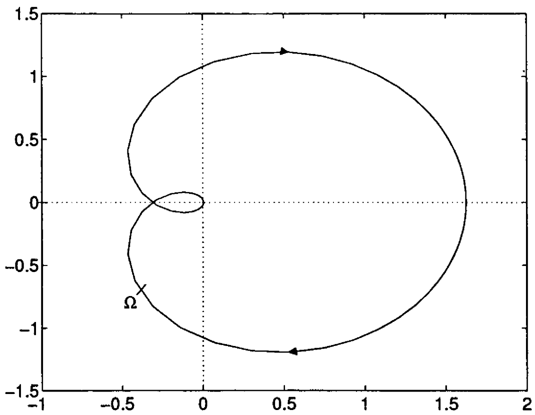

Fig. 3.3: Nyquist diagram of $m_1(s)m_2(s)$.

result in an increase to the other system's gain. That is, absolute feedback-loop gain is constant. Similarly, multipliers (or weights) cannot always be used to transform a feedback-loop such that the passivity theorem can be applied. There do exist transformations in the literature that transform a passive system to a system with gain less than one; and vice versa [19,64]. Using this idea, one could consider finding (frequency-dependent) transformations that transform "mixed" small gain and passive systems (as illustrated by systems with transfer functions $m_1(s)$ and $m_2(s)$) into either systems with small gain properties, or systems with passive properties, alone. These transformations would also have to preserve stability as far as the closed-loop goes. Initial investigations hint that such transformations in general may be difficult to find.

In this chapter, the idea of merging passivity and small gain concepts in the frequency domain for LTI systems is developed. This chapter considers causal, stable, MIMO systems, connected in a negative feedback-loop as illustrated in Fig. 3.4, where each system has associated with it a "mixed" small gain and passivity frequency domain property (demonstrated by the systems described by transfer functions $m_1(s)$ and $m_2(s)$ above). We exploit the notion of dissipativity, initiated by [93] and used by [40, 44, 69, 70] to produce stability results for interconnected systems, to mathematically describe the "mixed" small gain and passivity frequency domain property. This description is provided in Section 3.2. The main result of the chapter shows that finite-gain stability of the feedback interconnection is guaranteed. The feedback interconnection is described in Section 3.3 and the main stability result is given in Section 3.4. Section 3.5 concludes the chapter. The ideas in this chapter are presented as much for motivating nonlinear results of the same character (which are the subject of Chapter 4), as they are for their intrinsic interest.

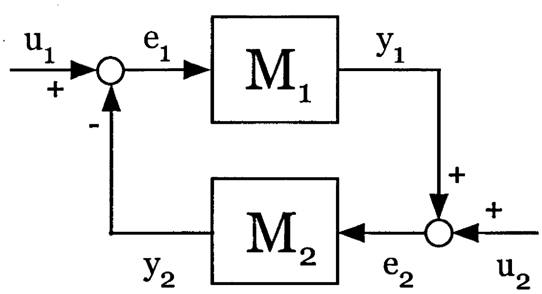

Fig. 3.4: Feedback interconnection of M_1 and M_2.

Preliminaries The results of this chapter are best viewed in the frequency domain. We consider frequency domain signals $f \in \mathcal{H}_2$, where $\mathcal{H}_2$ denotes the real frequency domain Hardy space in which

$$\|f\| = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega) f(j\omega) d\omega \right\}^{\frac{1}{2}}$$

and the superscript $(\cdot)^*$ denotes the complex conjugate transpose. $\mathcal{H}_2$ is a Hilbert space under the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^*(j\omega) f(j\omega) d\omega.$$

$\mathcal{R}$ denotes the set of proper real rational transfer function matrices. For a transfer function matrix $G \in \mathcal{R}$, $G^*(s)$ is defined to mean $G(-s)^T$. $\mathcal{L}_\infty$ is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on $j\mathbb{R}$. The Hardy space, $\mathcal{H}_\infty$, is the closed subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right half-plane (RHP), with norm denoted $\| \cdot \|_\infty$. In other words, $\mathcal{H}_\infty$ is the space of transfer functions of stable, LTI, continuous-time systems. $\mathcal{RH}_\infty$ denotes the subspace of $\mathcal{H}_\infty$ whose transfer functions are proper and real rational, and consequently, are analytic and bounded in the closed RHP.

3.2 System Descriptions

We want to formulate a mathematical description for a causal LTI system with transfer function matrix $M \in \mathcal{RH}_\infty$ that has the following frequency domain property. Consider the frequency range $-\infty < \omega < \infty$ and divide this range into intervals for which system M is: a) "input and output strictly passive"; b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one". This property will be referred to as the "mixed" small gain and passivity frequency domain property of a system M. What is meant by a system being input and output strictly passive on a frequency interval, and/or having gain less than one on a frequency interval, is defined below. The standard notions of input and output strict passivity and system gain which refer to the full $j\omega$-axis are also provided.

Definition 3. *[19, 82] Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$. This system is input and output strictly passive if $\exists l > 0$, $k > 0$ such that*

$$\langle Mx, x\rangle \geq l\|x\|^2 + k\|Mx\|^2 \tag{3.1}$$

$\forall x \in \mathcal{H}_2$. The system is said to be input strictly passive if (3.1) is satisfied with $k = 0$; output strictly passive if (3.1) is satisfied with $l = 0$; and passive if (3.1) is satisfied with $k = l = 0$.

In [44, 45, 70], input and output strict passivity is referred to as very strong passivity (VSP).

Definition 4. *Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$ and consider frequencies in the interval $[a, b]$. Call the system input and output strictly passive on the frequency interval $[a, b]$ if $\exists l > 0$, $k > 0$ such that*

$$\langle Mx, x\rangle_{[a,b]} \geq l\|x\|^2_{[a,b]} + k\|Mx\|^2_{[a,b]} \tag{3.2}$$

$\forall x \in \mathcal{H}_2$, where, given $x, y \in \mathcal{H}_2$,

$$\langle y, x\rangle_{[a,b]} := \frac{1}{2\pi}\int_a^b x^*(j\omega)y(j\omega)d\omega \tag{3.3}$$

and

$$\|(\cdot)\|^2_{[a,b]} := \langle(\cdot), (\cdot)\rangle_{[a,b]}. \tag{3.4}$$

The system is said to be input strictly passive on the frequency interval $[a, b]$ if (3.2) is satisfied with $k = 0$; output strictly passive on the frequency interval $[a, b]$ if (3.2) is satisfied with $l = 0$; and passive on the frequency interval $[a, b]$ if (3.2) is satisfied with $k = l = 0$.

Recall that a system with transfer function matrix in $\mathcal{RH}_\infty$ gives output in $\mathcal{H}_2$ whenever its input is in $\mathcal{H}_2$. We thus define the gain of the system as follows.

Definition 5. *Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$. The gain of the system is defined as*

$$\epsilon := \inf\{\bar{\epsilon} \in \mathbb{R}_+ : \|Mx\| \leq \bar{\epsilon}\|x\| \ \forall x \in \mathcal{H}_2\}.$$

If $\epsilon < 1$, then the system is said to have gain less than one; if $\epsilon \leq 1$, then the system is said to have gain less than or equal to one.

Definition 6. *Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$ and consider frequencies in the finite interval $[a, b]$. Define the system gain over the frequency interval $[a, b]$ to be*

$$\epsilon := \inf\{\bar{\epsilon} \in \mathbb{R}_+ : \|Mx\|_{[a,b]} \leq \bar{\epsilon}\|x\|_{[a,b]} \ \forall x \in \mathcal{H}_2\},$$

where $\|(\cdot)\|_{[a,b]}$ is defined by (3.4).

If $\epsilon < 1$, then the system is said to have gain less than one on the frequency interval $[a, b]$; if $\epsilon \leq 1$, then the system is said to have gain less than or equal to one on the frequency interval $[a, b]$.

Finite frequency intervals $[a, b]$ are considered in the above definitions of input and output strict passivity on a frequency interval and system gain on a frequency interval. However, infinite frequency intervals $[a, b)$, $(a, b]$ or (a, b), where a or b may be equal to $\pm\infty$, may be considered by taking improper integrals in (3.3) and (3.4) where appropriate.

The notion of dissipativity is used to describe the "mixed" small gain and passivity frequency domain property of a system M as follows. First we give a definition of a dissipative system.

Definition 7. *Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$. Denote the system's input and output signals, $e \in \mathcal{H}_2$ and $y \in \mathcal{H}_2$, respectively. The system is said to be dissipative with respect to the real triple $(Q(\omega), S(\omega), R(\omega))$ if*

$$\langle y, Q(\omega)y\rangle + 2\langle y, S(\omega)e\rangle + \langle e, R(\omega)e\rangle \geq 0$$

$\forall e \in \mathcal{H}_2$, where $Q(\omega)$ and $R(\omega)$ are self-adjoint at every ω (ie: $Q(\omega)^T = Q(\omega)$ and $R(\omega)^T = R(\omega)$) and $Q(\omega)$ is also negative semi-definite at every ω.

Define a real, continuous, (even) function of frequency that is: i) equal to one on frequency intervals for which M is considered "input and output strictly passive"; ii) equal to zero on frequency intervals for which M is considered to have "gain less than one"; and iii) is strictly greater than zero and strictly less than one on frequency intervals for which M is considered "input and output strictly passive with gain less than one". Denote this function $\alpha(\omega)$. Then the "mixed" small gain and passivity frequency domain property of system M can be described by letting

$$\begin{aligned}
Q_m(\omega) &:= Q(\omega) = -(k\alpha(\omega) + 1 - \alpha(\omega))I \\
S_m(\omega) &:= S(\omega) = \alpha(\omega)I \\
R_m(\omega) &:= R(\omega) = (\epsilon^2(1 - \alpha(\omega)) - l\alpha(\omega))I
\end{aligned}$$

in Definition 7, where $\epsilon < 1$, $l > 0$ and $k > 0$. The statement that system M is dissipative with respect to the triple $(Q_m(\omega), S_m(\omega), R_m(\omega))$ means that

$$\langle y, Q_m y\rangle + 2\langle y, S_m e\rangle + \langle e, R_m e\rangle \geq 0 \tag{3.5}$$

$\forall e \in \mathcal{H}_2$.

To see that the desired "mixed" property of a system M is accurately described using the notion of dissipativity as above, note that the left-hand side (LHS) of (3.5) is equal to

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} Q_m(\omega)e^*(j\omega)M^*(j\omega)M(j\omega)e(j\omega)d\omega \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty} S_m(\omega)e^*(j\omega)[M^*(j\omega)+M(j\omega)]e(j\omega)d\omega \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty} R_m(\omega)e^*(j\omega)e(j\omega)d\omega.
\end{aligned} \tag{3.6}$$

Let us continue by illustrating with a simple example. Suppose that system M has gain less than one on the frequency intervals $(-\infty, -\omega_b]$ and $[\omega_b, \infty)$; is input and output strictly passive and has gain less than one on the frequency intervals $(-\omega_b, -\omega_a)$ and (ω_a, ω_b); and is input and output strictly passive on the frequency interval $[-\omega_a, \omega_a]$. For instance, for the system described by the transfer function $m_1(s)$ in Section 3.1, we could take $\omega_a = 0.924$ and $\omega_b = 1.414$.

Breaking the integrals from $-\infty$ to ∞ of (3.6) into integrals from $-\infty$ to $-\omega_b$, $-\omega_b$ to $-\omega_a$, $-\omega_a$ to ω_a, ω_a to ω_b and ω_b to ∞; grouping the integrals from each respective frequency range together and adding the integrands; and substituting into the integrands values of $\alpha(\omega) = 1$ for the integrals from $-\omega_a$ to ω_a, and $\alpha(\omega) = 0$ for the integrals from $-\infty$ to $-\omega_b$ and ω_b to ∞, gives

$$\frac{1}{2\pi}\int_{\omega_b}^{\infty} e^*(\epsilon^2 I - M^*M)e\, d\omega \tag{3.7}$$

$$+\frac{1}{2\pi}\int_{\omega_a}^{\omega_b} e^*(Q_m M^*M + S_m(M^*+M) + R_m)e\, d\omega \tag{3.8}$$

$$+\frac{1}{2\pi}\int_{-\omega_a}^{\omega_a} e^*(M^*+M-kM^*M-lI)e\, d\omega \tag{3.9}$$

$$+\frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} e^*(Q_m M^*M + S_m(M^*+M) + R_m)e\, d\omega \tag{3.10}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{-\omega_b} e^*(\epsilon^2 I - M^*M)e\, d\omega. \tag{3.11}$$

Integrals (3.7) and (3.11) are greater than or equal to zero since M has gain less than one on the frequency intervals $(-\infty, -\omega_b]$ and $[\omega_b, \infty)$. Integral (3.9) is greater than or equal to zero since M is input and output strictly passive on the frequency interval $[-\omega_a, \omega_a]$. It remains to show that integrals (3.8) and (3.10) are greater than or equal to zero. Note that integral (3.8) is equal to

$$\frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \alpha e^*(M^* + M - kM^*M - lI)e\, d\omega + \frac{1}{2\pi}\int_{\omega_a}^{\omega_b}(1-\alpha)e^*(\epsilon^2 I - M^*M)e\, d\omega,$$

which is greater than or equal to zero because $0 \leq \alpha(\omega) \leq 1$ and M is both input and output strictly passive and has gain less than one on the frequency interval (ω_a, ω_b). Similarly, integral (3.10) is greater than or equal to zero.

We conclude the section with a comment on the division of the frequency range.

Remark 2. *The division of the frequency range $-\infty < \omega < \infty$ into intervals for which a system M is: a) "input and output strictly passive"; b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one" should be interpreted to mean to divide the frequency range $-\infty < \omega < \infty$ into intervals for which a system M is a) "input and output strictly passive" (and may or may not have "gain less than one"); b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one" (and may or may not be "input and output strictly passive").*

For example, consider Nyquist diagrams of transfer functions of SISO systems with the "mixed" small gain and passivity frequency domain property, such as $m_1(s)$ and $m_2(s)$ given in Section 3.1. Remark 2 indicates that it is not required that the divisions of the frequency range occur precisely at those frequencies for which the Nyquist diagrams cross the unit circle and the $j\omega$-axis. For example, the Nyquist diagram of $m_1(s)$ indeed crosses the unit circle at frequencies ± 0.924 and crosses the $j\omega$-axis at frequencies ± 1.414, and so one could take $\omega_a = 0.924$ and $\omega_b = 1.414$. However, the notion of using dissipativity to describe the "mixed" small gain and passivity frequency domain property of a system still holds if we take $1.414 > \omega_b > \omega_a > 0.924$.

The above comment is non-trivial in the following manner. Later, we will be interested in determining stability of negative feedback interconnections of two systems, where each system has a "mixed" small gain and passivity frequency domain property. To determine stability, we require that common frequency intervals can be found on which both systems in the feedback interconnection are "input and output strictly passive with gain less than one". That is, we require that the frequency range $-\infty < \omega < \infty$ can be divided into intervals for which the two systems in the interconnection are both: a) "input and output strictly passive" (and one or both of the systems may or may not have "gain less than one"); or b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one" (and one or both of the systems may or may not be "input and output strictly passive"). For instance, consider the interconnection of the two systems described by the transfer functions $m_1(s)$ and $m_2(s)$ given in Section 3.1. These systems are "input and output strictly passive with gain less than one" on, say, the common frequency intervals $(-1.4, -1)$ and $(1, 1.4)$ (as shown in Figs. 3.1 and 3.2). It would therefore be satisfactory to set $\omega_a = 1$ and $\omega_b = 1.4$.

Discussion on the relaxation of the requirement of "input and output strict passivity on a frequency interval" for one of the systems in the feedback-loop occurs later. This discussion is important because the situation is analogous to the passivity theorem's supposition that one system be input and output strictly passive, while the other system may simply be passive. We also discuss, by giving an example, how multipliers (or weights) can be used to scale the interconnection when one or both of the original systems in the feedback-loop do not exhibit the "mixed" small gain and passivity frequency domain property, hence increasing the system class size for which the results presented here are applicable.

3.3 The Feedback Interconnection

We now consider the feedback interconnection of two systems M_1 and M_2, as shown in Fig. 3.4, which are each dissipative in the sense of Definition 7 (keeping in mind Remark 2 and the comments made in the second last paragraph of the previous section). Let the $(Q(\omega), S(\omega), R(\omega))$ triple associated with system M_i, $i = 1, 2$, be given by

$$Q_i(\omega) = -(k_i\alpha(\omega) + 1 - \alpha(\omega))I \tag{3.12}$$

$$S_i(\omega) = \alpha(\omega)I \tag{3.13}$$

$$R_i(\omega) = (\epsilon_i^2(1 - \alpha(\omega)) - l_i\alpha(\omega))I \tag{3.14}$$

where $\alpha(\omega)$ is as described previously. In the spirit of [40, 44, 69, 70], where constant (Q_i, S_i, R_i) triples are considered as opposed to frequency-dependent triples, we show that the interconnected system is also dissipative (in a sense to be described). This description of dissipativity of the closed-loop provides us with a tool to prove finite-gain stability of the interconnection (which is realized in the next section).

We denote the interconnection of systems M_1 and M_2 by M_{sys}. So M_1 and M_2 are interconnected via

$$e_1 = u_1 - y_2 \tag{3.15}$$

$$e_2 = u_2 + y_1 \tag{3.16}$$

as indicated in Fig. 3.4. The input and output signal space for M_{sys} is the product space $\mathcal{H}_2 \times \mathcal{H}_2$, and the elements of the input and output signal space are $u := \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$ and $y := \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$, respectively. Note that inner products in these spaces are derived by summing inner products in the component spaces.

Assume that the system M_{sys} is well-posed in the sense of [104]. Write (3.15) and (3.16) in the compact form

$$e = u - Hy \tag{3.17}$$

where $H := \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Define

$$\begin{aligned}
\tilde{Q} &:= \begin{pmatrix} Q_1(\omega) & 0 \\ 0 & Q_2(\omega) \end{pmatrix} \\
\tilde{S} &:= \begin{pmatrix} S_1(\omega) & 0 \\ 0 & S_2(\omega) \end{pmatrix} \\
\tilde{R} &:= \begin{pmatrix} R_1(\omega) & 0 \\ 0 & R_2(\omega) \end{pmatrix}.
\end{aligned}$$

Then, similarly to [40,44,69,70], it can be shown that M_{sys} is $(\bar{Q}, \bar{S}, \tilde{R})$ dissipative, where

$$\begin{aligned}
\bar{Q} &:= \tilde{Q} + H^T\tilde{R}H - \tilde{S}H - H^T\tilde{S}^T \\
&= \begin{pmatrix} -\bar{q}_1 I & 0 \\ 0 & -\bar{q}_2 I \end{pmatrix}
\end{aligned}$$

with $\bar{q}_1 := (1-\epsilon_2^2)(1-\alpha(\omega)) + (k_1 + l_2)\alpha(\omega) > 0$, $\bar{q}_2 := (1-\epsilon_1^2)(1-\alpha(\omega)) + (k_2 + l_1)\alpha(\omega) > 0$ and

$$\begin{aligned}
\bar{S} &:= \tilde{S} - H^T\tilde{R} \\
&= \begin{pmatrix} \alpha(\omega)I & \bar{s}_1 I \\ -\bar{s}_2 I & \alpha(\omega)I \end{pmatrix}
\end{aligned}$$

with $\bar{s}_1 := \epsilon_2^2(1-\alpha(\omega)) - l_2\alpha(\omega)$, $\bar{s}_2 := \epsilon_1^2(1-\alpha(\omega)) - l_1\alpha(\omega)$, by adding inequalities

$$\langle y_i, Q_i y_i\rangle + 2\langle y_i, S_i e_i\rangle + \langle e_i, R_i e_i\rangle \geq 0$$

with $i = 1, 2$ and substituting (3.17) in as follows:

$$\begin{aligned}
&\langle y_1, Q_1 y_1\rangle + \langle y_2, Q_2 y_2\rangle + 2\langle y_1, S_1 e_1\rangle + 2\langle y_2, S_2 e_2\rangle + \langle e_1, R_1 e_1\rangle + \langle e_2, R_2 e_2\rangle \geq 0 \\
&\Leftrightarrow \langle y, \tilde{Q}y\rangle + 2\langle y, \tilde{S}e\rangle + \langle e, \tilde{R}e\rangle \geq 0 \\
&\Leftrightarrow \langle y, \tilde{Q}y\rangle + 2\langle y, \tilde{S}u - \tilde{S}Hy\rangle + \langle u - Hy, \tilde{R}u - \tilde{R}Hy\rangle \geq 0 \\
&\Leftrightarrow \langle y, \tilde{Q}y\rangle + \langle y, H^T\tilde{R}Hy\rangle + \langle y, -\tilde{S}Hy\rangle + \langle y, -H^T\tilde{S}^T y\rangle + 2\langle y, \tilde{S}u\rangle
\end{aligned}$$

$$+\,2\langle y, -H^T\tilde{R}u\rangle + \langle u, \tilde{R}u\rangle \geq 0$$
$$\Leftrightarrow \langle y, \bar{Q}y\rangle + 2\langle y, \bar{S}u\rangle + \langle u, \tilde{R}u\rangle \geq 0.$$

3.4 Main Stability Theorem

It is now shown that input-output stability of the interconnected system M_{sys} (as described in the previous section) is always guaranteed.

Theorem 3. *Consider two causal systems with transfer function matrices $M_1 \in \mathcal{RH}_\infty$ and $M_2 \in \mathcal{RH}_\infty$ which are interconnected as shown in Fig. 3.4. Furthermore, suppose that systems M_1 and M_2 are dissipative in the sense of Definition 7 with respect to the triples $(Q_i(\omega), S_i(\omega), R_i(\omega))$, $i = 1, 2$, given at the beginning of Section 3.3. Then the interconnection of the systems, denoted M_{sys}, is finite-gain stable.*

Proof. Note that $\bar{\bar{Q}} := -\bar{Q}$ is positive definite. As in [40,44,69,70], but considering frequency-dependent (as opposed to constant) $\bar{\bar{Q}}$, it is shown that, since $\bar{\bar{Q}}$ is positive definite, M_{sys} is finite-gain stable.

From Definition 7, the statement that M_{sys} is $(\bar{Q}, \bar{S}, \tilde{R})$ dissipative means that

$$\langle y, \bar{\bar{Q}}y\rangle - 2\langle y, \bar{\bar{Q}}^{\frac{1}{2}}\bar{\bar{S}}u\rangle \leq \langle u, \tilde{R}u\rangle$$

$\forall u \in \mathcal{H}_2$, where $\bar{\bar{S}} := \bar{\bar{Q}}^{-\frac{1}{2}}\bar{S}$. The matrix $\tilde{R} + \bar{\bar{S}}^T\bar{\bar{S}}$ is a symmetric matrix, equal to

$$\begin{pmatrix} (\bar{s}_2 + \frac{\alpha^2}{\bar{q}_1} + \frac{\bar{s}_2^2}{\bar{q}_2})I & \alpha(\omega)(\frac{\bar{s}_1}{\bar{q}_1} - \frac{\bar{s}_2}{\bar{q}_2})I \\ \alpha(\omega)(\frac{\bar{s}_1}{\bar{q}_1} - \frac{\bar{s}_2}{\bar{q}_2})I & (\bar{s}_1 + \frac{\bar{s}_1^2}{\bar{q}_1} + \frac{\alpha^2}{\bar{q}_2})I \end{pmatrix}.$$

Then $\tilde{R} + \bar{\bar{S}}^T\bar{\bar{S}}$ is orthogonally similar to a diagonal matrix, ie:

$$\tilde{R}(\omega) + \bar{\bar{S}}(\omega)^T\bar{\bar{S}}(\omega) = U(\omega)^T D(\omega) U(\omega),$$

and so there always exists a finite scalar $\kappa > 0$ such that $\tilde{R} + \bar{\bar{S}}^T\bar{\bar{S}} \leq \kappa^2 I$, ie: $U(\omega)^T D(\omega) U(\omega) \leq \kappa^2 I = \kappa^2 U(\omega)^T U(\omega)$ and

$$U(\omega)^T \begin{pmatrix} \lambda_1(\omega) - \kappa^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_p(\omega) - \kappa^2 \end{pmatrix} U(\omega) \leq 0.$$

So $\exists \kappa > 0$ such that

$$\langle y, \bar{\bar{Q}}y\rangle - 2\langle y, \bar{\bar{Q}}^{\frac{1}{2}}\bar{\bar{S}}u\rangle \leq \kappa^2\langle u, u\rangle - \langle u, \bar{\bar{S}}^T\bar{\bar{S}}u\rangle \tag{3.18}$$

$\forall u \in \mathcal{H}_2$.

Inequality (3.18) is equivalent to

$$\begin{aligned}&\langle y, \bar{\bar{Q}}^{\frac{1}{2}}\bar{\bar{Q}}^{\frac{1}{2}}y\rangle - 2\langle y, \bar{\bar{Q}}^{\frac{1}{2}}\bar{\bar{S}}u\rangle + \langle u, \bar{\bar{S}}^T\bar{\bar{S}}u\rangle \leq \kappa^2\langle u, u\rangle \\ &\Leftrightarrow \|\bar{\bar{Q}}^{\frac{1}{2}}y - \bar{\bar{S}}u\|^2 \leq \kappa^2\|u\|^2 \\ &\Leftrightarrow \|\bar{\bar{Q}}^{\frac{1}{2}}y - \bar{\bar{S}}u\| \leq \kappa\|u\|.\end{aligned}$$

It follows easily that

$$\|\bar{\bar{Q}}^{\frac{1}{2}}y\| \leq (\kappa + \|\bar{\bar{S}}\|_\infty)\|u\|. \tag{3.19}$$

Finally, note that $y = (\bar{\bar{Q}}^{\frac{1}{2}})^{-1}\bar{\bar{Q}}^{\frac{1}{2}}y$ implies that $\|y\| \leq \|\bar{\bar{Q}}^{-\frac{1}{2}}\|_\infty\|\bar{\bar{Q}}^{\frac{1}{2}}y\|$, or $\|\bar{\bar{Q}}^{-\frac{1}{2}}\|_\infty^{-1}$ $\|y\| \leq \|\bar{\bar{Q}}^{\frac{1}{2}}y\|$. Then, from (3.19),

$$\begin{aligned}&\|\bar{\bar{Q}}^{-\frac{1}{2}}\|_\infty^{-1}\|y\| \leq (\kappa + \|\bar{\bar{S}}\|_\infty)\|u\| \\ &\Leftrightarrow \|y\| \leq \bar{k}\|u\|,\end{aligned}$$

where $\bar{k} := \|\bar{\bar{Q}}^{-\frac{1}{2}}\|_\infty(\kappa + \|\bar{\bar{S}}\|_\infty)$. □

That is, by setting the $(Q(\omega), S(\omega), R(\omega))$ triples associated with systems M_1 and M_2 to be equal to the triples given by (3.12), (3.13) and (3.14), mathematical descriptions in terms of dissipativity can be given to describe the "mixed" small gain and passivity frequency domain property of each of the systems. With respect to the interconnection of the two systems, these mathematical descriptions allow for the frequency range $-\infty < \omega < \infty$ to be divided into intervals for which both systems M_1 and M_2 are: a) simultaneously "input and output strictly passive" (and may or may not have "gain less than one"); b) simultaneously "input and output strictly passive and with gain less than one"; or c) simultaneously "with gain less than one" (and may or may not be "input and output strictly passive"). Given the dissipative property of systems M_1 and M_2, it was shown that the interconnected system M_{sys} is $(\bar{Q}, \bar{S}, \bar{R})$ dissipative; and since $\bar{Q}$ is negative definite, then M_{sys} is finite-gain stable.

Suppose we let k_1 and k_2 from (3.12) be equal to zero. (We could say that this corresponds to relaxing input and output strict passivity on a frequency interval, to input strict passivity on a frequency interval.) Note that $\bar{Q}$ remains negative definite and so finite-gain stability of M_{sys} is still guaranteed. Alternatively, let l_1 and l_2 from (3.14) be equal to zero (which we could say corresponds to relaxing input and output strict passivity on a frequency interval, to output strict passivity on a frequency interval). In this case, $\bar{Q}$ also remains negative definite and so finite-gain stability of M_{sys} is still guaranteed. Alternatively still, let k_1 and l_1 (or k_2 and l_2) of (3.12) and (3.14) be equal to zero (which corresponds to relaxing input and output strict passivity on a frequency interval of system M_1 (or system M_2), to passivity on a frequency interval). The matrix $\bar{Q}$ remains negative definite, and so finite-gain stability of M_{sys} is guaranteed in this case also.

Now we discuss how multipliers (or weights) can be used to scale the interconnection when one or both of the original systems in the feedback-loop do not exhibit the "mixed" small gain and passivity frequency domain property. Appropriate scaling of the interconnection so that both weighted systems are of the "mixed" small gain and passive type increases the system class size for which the results of this chapter are applicable. For example, let us consider the systems with transfer functions

$$n_1(s) = \frac{1}{s^2 + 0.2s + 1}$$

and

$$n_2(s) = \frac{10}{\frac{1}{0.009}s + 1}$$

with Nyquist diagrams shown in Figs. 3.5 and 3.6. Clearly, the system with transfer function $n_1(s)$ is neither "input and output strictly passive", nor has "gain less than one", on the frequency interval between 1 rad/s and 1.4 rad/s. Yet the feedback interconnection of the two systems is stable, as shown by the Nyquist diagram of $n_1(s)n_2(s)$, illustrated in Fig. 3.7, as it does not encircle the point $-1 + j0$.

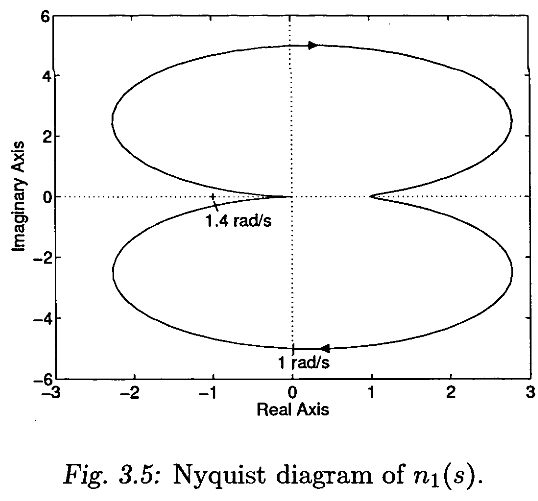

Fig. 3.5: Nyquist diagram of $n_1(s)$.

Suppose that we introduce the constant multiplier $\gamma = 0.1$ into the feedback interconnection. That is, let us scale the interconnection by (pre-) multiplying $n_1(s)$ by γ and (post-)multiplying $n_2(s)$ by γ^{-1} to give

$$\gamma n_1(s) = \frac{0.1}{s^2 + 0.2s + 1}$$

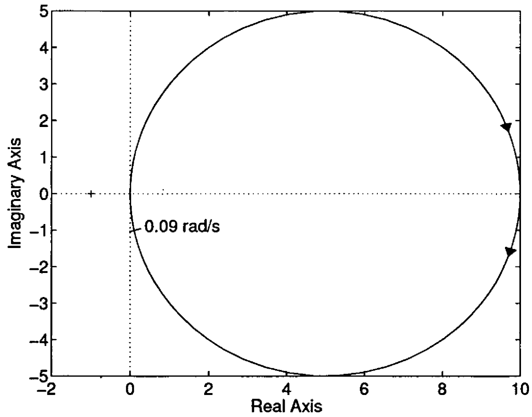

Fig. 3.6: Nyquist diagram of $n_2(s)$.

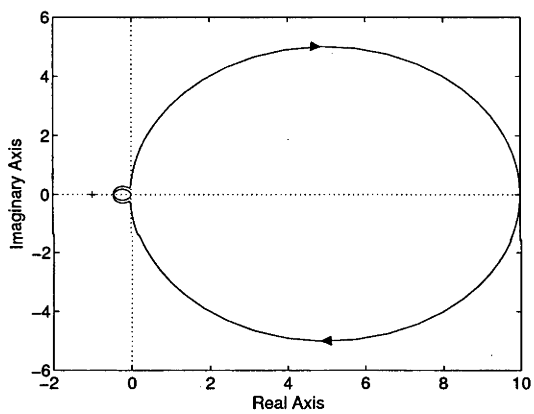

Fig. 3.7: Nyquist diagram of $n_1(s)n_2(s)$.

and

$$n_2(s)\gamma^{-1} = \frac{100}{\frac{1}{0.009}s + 1}$$

with Nyquist diagrams shown in Figs. 3.8 and 3.9. It is clear that both of these scaled systems, ie: the systems with transfer functions $\gamma n_1(s)$ and $n_2(s)\gamma^{-1}$, do have the "mixed" small gain and passivity frequency domain property. Furthermore, it is possible to find a common frequency interval on which both scaled systems are "input and output strictly passive and with gain less than one" (we could choose the frequency interval $(0.91, 0.99)$ for example). In effect, we can apply the techniques of this chapter to guarantee stability of the interconnection of the scaled systems, and since multiplier theory preserves stability, stability of the original feedback interconnection can be inferred, as expected.

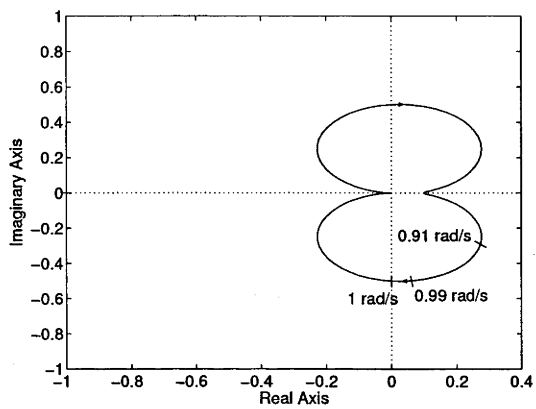

Fig. 3.8: Nyquist diagram of $\gamma n_1(s)$.

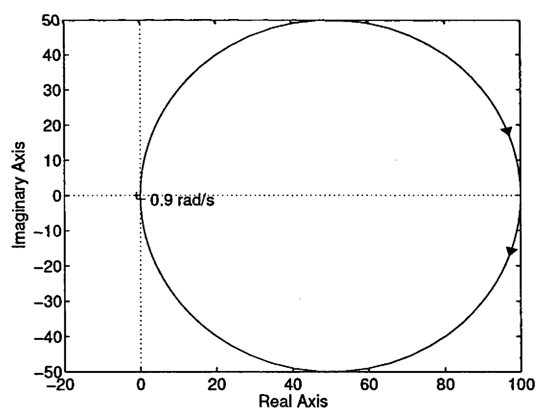

Fig. 3.9: Nyquist diagram of $n_2(s)\gamma^{-1}$.

Generally, one chooses weights which are units in $\mathcal{RH}_\infty$. Then the modified feedback interconnection consists of the scaled systems $W_1M_1W_2^{-1}$ and $W_2M_2W_1^{-1}$ replacing M_1 and M_2, respectively, where W_1 and W_2 represent the weights.

3.5 Conclusions

It was shown that finite-gain stability is guaranteed for a feedback-loop (denoted M_{sys}) that consists of two causal, stable, LTI systems, M_1 and M_2, where each system has a "mixed" small gain and passivity frequency domain property associated with it. This property was described via the notion of dissipative systems. It is clear that, in the case of MIMO LTI systems, there already exist simple techniques to determine stability of a feedback interconnection. For example, one needs only to check that the transfer function matrix mapping signals u_1 and u_2 to e_1 and e_2 of Fig. 3.4 are in $\mathcal{RH}_\infty$. However, these simple techniques often fail in the time-varying and/or nonlinear case. It is desired that a stability result for systems with "mixed" small gain and passivity properties can be provided in the time-varying and/or nonlinear case. Such a result is the subject of the next chapter.

4. INPUT-OUTPUT STABILITY RESULTS FOR NONLINEAR SYSTEMS WITH "MIXED" PROPERTIES[1]

4.1 Introduction

In the previous chapter, a frequency domain stability result for the feedback interconnection of two stable LTI systems was provided. The assumption placed on the LTI systems was that they both exhibited a "mixed" small gain and passivity frequency domain property. In this chapter, we wish to describe a "mixed" small gain and passivity property for causal, nonlinear systems in the time domain, and provide the associated feedback interconnection input-output stability results.

Let us recall the notion of the "mixed" small gain and passivity frequency domain property. Consider the negative feedback interconnection of two SISO LTI systems with transfer functions $\hat{m}_1(s) = \frac{3}{(s+1)(s+2)}$ and $\hat{m}_2(s) = \frac{13}{(s+3)(s+4)}$, as was discussed in Chapter 3. The Nyquist diagrams of these transfer functions were shown in Figs. 3.1 and 3.2, respectively. Since the product of the gains of the two systems is greater than one, the small gain theorem cannot be used as a tool to determine stability; and since the systems are not passive, stability cannot be guaranteed via the passivity theorem.

Notice that there exists a common frequency interval over which both systems are "passive" (and may or may not have "gain less than one"); and a common frequency interval over which both systems have "small gain" (and may or may not be "passive"). Systems exhibiting such "mixed" properties were mathematically described as follows: there exist constants $0 \leq \epsilon < 1$, $k > 0$ and $l > 0$ such that

$$-\langle \hat{m}\hat{f}, (k\alpha + 1 - \alpha)\hat{m}\hat{f}\rangle + 2\langle \hat{m}\hat{f}, \alpha\hat{f}\rangle - \langle \hat{f}, (l\alpha - \epsilon(1-\alpha))\hat{f}\rangle \geq 0 \qquad (4.1)$$

$\forall \hat{f} \in \mathcal{H}_2$; where $\hat{m} \in \mathcal{RH}_\infty$ and $\alpha(\omega)$ is a real, continuous, even function of frequency that is: i) equal to one on frequency intervals for which the system described by $\hat{m}(s)$ is "input and output strictly passive"; ii) equal to zero on frequency intervals for which the system described by $\hat{m}(s)$ has "gain less than one"; and iii) is strictly

[1] Results of the nature of those appearing in this chapter were submitted to *46th IEEE Conference on Decision and Control* [36]; and to *Systems & Control Letters* [33].

greater than zero and strictly less than one on frequency intervals for which the system described by $\hat{m}(s)$ is "input and output strictly passive with gain less than one". (The hat notation has been introduced to denote objects associated with the frequency domain.) This description captures the standard frequency domain concepts of passivity and small gain; it also captures a concept of "blending" of the passivity and small gain notions. In other words, the description captures a class of systems that is larger than the class of passive systems together with the class of systems with small gain.

Before defining a "mixed" small gain and passivity time domain property for nonlinear systems, which then leads us to the main business of the chapter of providing the associated input-output feedback interconnection stability result, let us derive a time domain analog of the LTI "mixed" small gain and passivity frequency domain property. This will help to motivate the definition of the "mixed" property for nonlinear systems. We must first establish some preliminary mathematics and notation (some of which has been repeated from Chapter 2 for convenience). The field of real numbers is denoted by $\mathbb{R}$. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are real inner product spaces. The inner product of $\mathcal{X}$ is denoted by $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$. A norm for each element of $\mathcal{X}$ is defined to be $\|f\|_{\mathcal{X}}^2 = \langle f, f \rangle$. An important property of inner product spaces is the so-called Cauchy-Schwarz inequality; that is $|\langle f, g \rangle| \leq \|f\|_{\mathcal{X}} \|g\|_{\mathcal{X}}$ $\forall f, g \in \mathcal{X}$. Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. For a bounded linear operator $H : \mathcal{H} \to \mathcal{K}$, the Hilbert adjoint $H^{\sim} : \mathcal{K} \to \mathcal{H}$ of H is defined by $\langle Hh, k \rangle = \langle h, H^{\sim} k \rangle$ for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$.

Let $\mathcal{L}_2[0, \infty)$ denote the Lebesgue space with inner product defined as

$$\langle f, g \rangle = \int_0^\infty g'(t) f(t) dt,$$

where the superscript $(\cdot)'$ denotes the vector transpose. In this chapter, the norm of functions in $\mathcal{L}_2[0, \infty)$ is denoted by $\| \cdot \|$. For $T \in [0, \infty)$, let P_T denote the truncation operator. That is, for a function $f(t)$, $0 \leq t < \infty$,

$$(P_T f)(t) := \begin{cases} f(t), & t \leq T \\ 0, & t > T \end{cases}.$$

For convenience, the notation $f_T := P_T f$ will be used. We define $\langle f, g \rangle_T := \langle f_T, g_T \rangle$ and note that $\langle f_T, g_T \rangle = \langle f_T, g \rangle = \langle f, g_T \rangle$. Let $\mathcal{L}_{2e}$ denote the extension of the space $\mathcal{L}_2[0, \infty)$, defined by $\mathcal{L}_{2e} := \{ f : f_T \in \mathcal{L}_2[0, \infty) \ \forall T \in [0, \infty) \}$. Recall that the space $\mathcal{L}_2[0, \infty)$ satisfies the following properties:

i) The space $\mathcal{L}_2[0, \infty)$ is such that if $f \in \mathcal{L}_2[0, \infty)$, then $f_T \in \mathcal{L}_2[0, \infty)$ $\forall T \in [0, \infty)$; and moreover, the space $\mathcal{L}_2[0, \infty)$ is such that $f = \lim_{T \to \infty} f_T$. Equivalently, the space $\mathcal{L}_2[0, \infty)$ is closed under the family of projections $\{P_T\}$.

ii) If $f \in \mathcal{L}_2[0,\infty)$ and $T \in [0,\infty)$, then $\|f_T\| \leq \|f\|$. Moreover, $\|f_T\|$ is a nondecreasing function of $T \in [0,\infty)$.

iii) If $f \in \mathcal{L}_{2e}$, then $f \in \mathcal{L}_2[0,\infty)$ if and only if $\lim_{T\to\infty} \|f_T\| < \infty$.

The term *system* will be used to refer to a mapping from $\mathcal{L}_{2e}$ into $\mathcal{L}_{2e}$, which satisfies a causality condition. An operator $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is causal if $P_T M P_T = P_T M$ for all $T \in [0,\infty)$. An operator $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is anticausal if $(I - P_T)M(I - P_T) = (I - P_T)M$ for all $T \in [0,\infty)$. A system mapping $\mathcal{L}_{2e}$ into $\mathcal{L}_{2e}$ is input-output $\mathcal{L}_2$-stable if the output belongs to $\mathcal{L}_2[0,\infty)$ whenever the input belongs to $\mathcal{L}_2[0,\infty)$. For simplicity, input-output $\mathcal{L}_2$-stability will be referred to as input-output stability, or stability, when the context is clear. It is assumed that all systems considered are relaxed systems (that is, they have zero initial state). The operator $I : \mathcal{X} \to \mathcal{X}$, defined by $Ix := x$ for all $x \in \mathcal{X}$, denotes the identity operator. The operator $\mathbf{0} : \mathcal{X} \to \mathcal{Y}$, defined by $\mathbf{0}x := 0$ for all $x \in \mathcal{X}$ (where 0 denotes the zero vector from $\mathcal{Y}$), denotes the zero operator.

We now provide the following brief description of a (not necessarily finite dimensional) LTI system (in the context of the input-output theory of systems) which may be found in texts such as [16, 19, 64, 84] and in [10, 11]. The discussion is limited to SISO systems for simplicity. Let $\mathcal{A}$ denote the set of generalized functions of the form

$$m(t) = \begin{cases} m_0\delta(t) + m_a(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where $m_0 \in \mathbb{R}$, $\delta(\cdot)$ denotes the unit impulse, and $m_a(\cdot)$ is such that

$$\int_0^\infty |m_a(\tau)| d\tau < \infty.$$

Let $\hat{\mathcal{A}}$ denote the set consisting of all functions that are Laplace transforms of elements of $\mathcal{A}$. An LTI system $M : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ is defined to be a convolution operator of the form

$$(Mf)(t) = m(t) * f(t) = \int_{-\infty}^{\infty} m(\tau)f(t-\tau)d\tau = \int_{-\infty}^{\infty} m(t-\tau)f(\tau)d\tau \qquad (4.2)$$

where $m(\cdot) \in \mathcal{A}$ [19, Section D.1]. The function $m(\cdot)$ is called the kernel, or the impulse response, of the operator M. Furthermore, since $m(\tau) = 0$ for $\tau < 0$ and $f(t) = 0$ for $t < 0$, from (4.2)

$$(Mf)(t) = m_0 f(t) + \int_0^t m_a(\tau)f(t-\tau)d\tau = m_0 f(t) + \int_0^t m_a(t-\tau)f(\tau)d\tau.$$

Then $\hat{m}(j\omega)$ as in (4.1) is the Fourier transform of $m(t)$; and let $\hat{f}(j\omega)$ denote the Fourier transform of input signal $f(t)$.

Suppose we introduce causal, bounded, linear operators $\Gamma : \mathcal{L}_2[0,\infty) \rightarrow \mathcal{L}_2[0,\infty)$ and $\mathrm{B} : \mathcal{L}_2[0,\infty) \rightarrow \mathcal{L}_2[0,\infty)$, where

$$\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B} = I. \tag{4.3}$$

Furthermore, suppose that Γ and B are time-invariant operators. Let $\gamma(\cdot)$ and $\beta(\cdot)$ denote the kernels of Γ and B, respectively, such that $\gamma(\cdot), \beta(\cdot) \in \mathcal{A}$. If $h^a(t) := h(-t)$ denotes the kernel of an anticausal LTI system, then

$$\gamma^a(t) * \gamma(t) + \beta^a(t) * \beta(t) = \delta(t) \tag{4.4}$$

from (4.3). (Recall that, if H is a linear causal operator, then its adjoint $H^{\sim}$ is anticausal [18].) Let $\hat{\gamma}(j\omega)$ and $\hat{\beta}(j\omega)$ denote the Fourier transforms of $\gamma(t)$ and $\beta(t)$, respectively. Then $\hat{\gamma}(-j\omega)\hat{\gamma}(j\omega) + \hat{\beta}(-j\omega)\hat{\beta}(j\omega) = 1$, since the kernel of the adjoint of a linear (causal) system is obtained by replacing $j\omega$ by $-j\omega$ when the kernel is expressed in terms of its Fourier transform. For convenience, let $(\cdot)^*(j\omega) := (\cdot)(-j\omega)$.

Now that we have defined $\hat{\gamma}(j\omega)$ and $\hat{\beta}(j\omega)$, we return to (4.1) and set $\alpha(\omega) = \hat{\beta}^*(j\omega)\hat{\beta}(j\omega)$. Rewriting (4.1) gives

$$-\langle \hat{m}\hat{f}, (k\hat{\beta}^*\hat{\beta} + \hat{\gamma}^*\hat{\gamma})\hat{m}\hat{f}\rangle + 2\langle \hat{m}\hat{f}, \hat{\beta}^*\hat{\beta}\hat{f}\rangle - \langle \hat{f}, (l\hat{\beta}^*\hat{\beta} - \epsilon\hat{\gamma}^*\hat{\gamma})\hat{f}\rangle \geq 0,$$

which is identical to

$$-\langle \hat{m}\hat{f}, \hat{\gamma}^*\hat{\gamma}\hat{m}\hat{f}\rangle + \epsilon\langle \hat{f}, \hat{\gamma}^*\hat{\gamma}\hat{f}\rangle - k\langle \hat{m}\hat{f}, \hat{\beta}^*\hat{\beta}\hat{m}\hat{f}\rangle + 2\langle \hat{m}\hat{f}, \hat{\beta}^*\hat{\beta}\hat{f}\rangle - l\langle \hat{f}, \hat{\beta}^*\hat{\beta}\hat{f}\rangle \geq 0.$$

Via the Paley-Wiener theorem [16, Theorem A.6.21], we can write

$$\begin{aligned}&-\langle Mf, \Gamma^{\sim}\Gamma Mf\rangle + \epsilon\langle f, \Gamma^{\sim}\Gamma f\rangle - k\langle Mf, \mathrm{B}^{\sim}\mathrm{B}Mf\rangle + 2\langle Mf, \mathrm{B}^{\sim}\mathrm{B}f\rangle \\ &- l\langle f, \mathrm{B}^{\sim}\mathrm{B}f\rangle \geq 0,\end{aligned}$$

which is identical to

$$\begin{aligned}&-\langle \Gamma Mf, \Gamma Mf\rangle + \epsilon\langle \Gamma f, \Gamma f\rangle - k\langle \mathrm{B}Mf, \mathrm{B}Mf\rangle + 2\langle \mathrm{B}Mf, \mathrm{B}f\rangle \\ &- l\langle \mathrm{B}f, \mathrm{B}f\rangle \geq 0.\end{aligned} \tag{4.5}$$

Inequality (4.5) provides us with a time domain version of the LTI "mixed" small gain and passivity property: given the existence of causal, bounded, LTI operators $\Gamma : \mathcal{L}_2[0,\infty) \rightarrow \mathcal{L}_2[0,\infty)$ and $\mathrm{B} : \mathcal{L}_2[0,\infty) \rightarrow \mathcal{L}_2[0,\infty)$ such that (4.3) is satisfied, a causal LTI system $M : \mathcal{L}_2[0,\infty) \rightarrow \mathcal{L}_2[0,\infty)$ has a "mixed" small gain and passivity property associated with it if there exist constants $0 \leq \epsilon < 1$, $k > 0$ and $l > 0$ such that (4.5) holds for all $f \in \mathcal{L}_2[0, \infty)$.

The focus of this chapter is now to define a "mixed" small gain and passivity property, in the time domain, for causal nonlinear systems, and to prove the associated input-output feedback interconnection stability result. Inequality (4.5) will be used as a guide at the intuitive level. The chapter is broken down into the following sections. In Section 4.2, the feedback interconnection under consideration is formally described. In Section 4.3, a "mixed" small gain and passivity property for a causal nonlinear system is defined. The feedback interconnection stability result is provided in Section 4.4. Conclusions are provided in Section 4.5.

4.2 Feedback System Description

We wish to derive an input-output stability result concerning the feedback interconnection shown in Fig. 4.1. This feedback interconnection is described by the equations

$$
\begin{aligned}
e_1 &= u_1 - y_2 & \qquad y_1 &= M_1 e_1 \\
e_2 &= u_2 + y_1 & \qquad y_2 &= M_2 e_2
\end{aligned}
$$

where $u_1, u_2 \in \mathcal{L}_{2e}$ are the (external) input signals; $e_1, e_2 \in \mathcal{L}_{2e}$ are the error signals; and $y_1, y_2 \in \mathcal{L}_{2e}$ are the output signals. The operators M_1 and M_2 are assumed to causally map $\mathcal{L}_{2e}$ into $\mathcal{L}_{2e}$. Furthermore, M_1 and M_2 each have associated with them a "mixed" small gain and passivity property (defined formally in the next section).

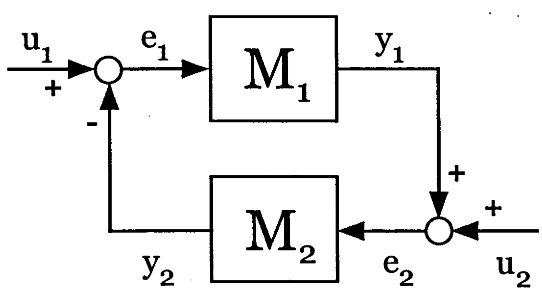

Fig. 4.1: Interconnection of M_1 and M_2.

Strictness and non-strictness of the "mixed" small gain and passivity property will be dealt with formally in later sections. Similarly to the passivity and small gain theorems, one of the systems in the feedback interconnection is required to have a strict form of the "mixed" small gain and passivity property associated with it.

Well-posedness of the feedback interconnection corresponds to the existence and uniqueness of solutions e_1, e_2 and y_1, y_2 for each choice of u_1, u_2; and furthermore, requires that e_1, e_2 and y_1, y_2 depend causally on u_1, u_2 [53]. It is usual to also require that e_1, e_2 and y_1, y_2 depend, on finite intervals, Lipschitz continuously on u_1, u_2 (as indicated in Section 1.1) [84]. References [39, 95] describe conditions to impose on the operators M_1 and M_2 to guarantee well-posedness of the feedback-loop. We do not discuss well-posedness further in this chapter; well-posedness of the feedback interconnection under consideration is assumed.

4.3 The "Mixed" Small Gain and Passivity Property

We seek to formally define what we refer to as the "mixed" small gain and passivity property associated with a system. As mentioned previously, the "mixed" small gain and passivity property can be thought of as a "blending" of the concepts of passivity and small gain. The concepts of finite gain and passivity are defined for nonlinear systems, in the time domain, below.

Definition 8. *[64] A system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to have a finite gain if there exist constants $\bar{\epsilon} \geq 0$ and $\eta \geq 0$, such that*

$$\|(Mf)_T\| \leq \bar{\epsilon}\|f_T\| + \eta \tag{4.6}$$

for all input signals $f \in \mathcal{L}_{2e}$ and all $T \in [0, \infty)$.

The constant η is called the bias term and is included to allow for the case where $Mf \neq 0$ when $f = 0$ [64]. Clearly, if there do exist constants $\bar{\epsilon}$ and η such that (4.6) holds, then $\bar{\epsilon}$ is not uniquely defined. We call the gain of M the number ϵ defined by

$$\epsilon = \inf\{\bar{\epsilon} \in \mathbb{R}_+ : \exists \eta \text{ such that inequality (4.6) holds}\}$$

(see [19, Section III.2]). If $\epsilon < 1$, then the system M is said to have gain less than one; if $\epsilon \leq 1$, then M is said to have gain less than or equal to one. Systems with finite gain are said to be finite-gain stable [64]. Obviously, if a system has finite gain, then the system is input-output stable.

Definition 9. *[64] A system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to be input and output strictly passive if there exist constants $k, l > 0$ such that*

$$\langle Mf, f\rangle_T \geq k\|(Mf)_T\|^2 + l\|f_T\|^2 \tag{4.7}$$

for all input signals $f \in \mathcal{L}_{2e}$ and all $T \in [0, \infty)$. The system M is said to be input strictly passive if it satisfies (4.7) with $k = 0$; output strictly passive if it satisfies (4.7) with $l = 0$; and, passive if it satisfies (4.7) with $k = l = 0$.

Note that input and output strict passivity is equivalent to input strict passivity with finite gain [45, 70, 82]. The (strict version of the) "mixed" small gain and passivity property is now defined below.

Definition 10. *Let $\Gamma : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ and $B : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ be causal, bounded, linear (and not necessarily time-invariant) operators such that*

$$\Gamma^{\sim}\Gamma + B^{\sim}B = I. \tag{4.8}$$

Then a system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to have a strict "mixed" small gain and passivity property if there exist constants $0 \leq \epsilon < 1$, $k > 0$, $l > 0$ and $\eta \geq 0$ such that

$$\begin{aligned} &- \langle \Gamma(Mf)_T, \Gamma(Mf)_T \rangle + \epsilon \langle \Gamma f_T, \Gamma f_T \rangle - k \langle B(Mf)_T, B(Mf)_T \rangle \\ &+ 2\langle B(Mf)_T, Bf_T \rangle - l \langle Bf_T, Bf_T \rangle + \eta \geq 0 \end{aligned} \tag{4.9}$$

for all input signals $f \in \mathcal{L}_{2e}$ and all $T \in [0,\infty)$.

The term η has been included to allow for output bias (that is, when zero system input does not imply zero system output) [69]. The "mixed" small gain and passivity property captures the concepts of passivity or small gain normally associated with a system. If $\Gamma = \mathbf{0}$, then (4.9) describes an input and output strictly passive system. If $\mathrm{B} = \mathbf{0}$, then (4.9) describes a system with gain less than one. The description of the "mixed" small gain and passivity property additionally captures a concept of "blending" of the small gain and passivity ideas. In the case of LTI M, Γ and B for example, if Γ is time-invariant with $|\gamma(j\omega)|$ close to 0 at low frequencies and close to 1 at high frequencies, then the mixed property in qualitative terms corresponds to the system being passive at low frequencies and having small gain at high frequencies. (Recall that Chapter 3 extensively illustrated the concept of "blending" of the small gain and passivity ideas in the frequency domain.)

In fact, the following observation can be made in regards to LTI systems. Suppose that the causal, bounded, linear operators $\Gamma, \mathrm{B} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ are time-invariant. Consider a causal, LTI system $M : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$. In this case, if M satisfies condition (4.9), then M also satisfies condition (4.5). To see this, first note that since M is linear (and has zero initial state), there is no loss in generality in setting $\eta = 0$ in (4.9) [41], giving

$$\begin{aligned} &- \langle \Gamma(Mf)_T, \Gamma(Mf)_T \rangle + \epsilon \langle \Gamma f_T, \Gamma f_T \rangle - k \langle \mathrm{B}(Mf)_T, \mathrm{B}(Mf)_T \rangle \\ &+ 2\langle \mathrm{B}(Mf)_T, \mathrm{B}f_T \rangle - l \langle \mathrm{B}f_T, \mathrm{B}f_T \rangle \geq 0. \end{aligned} \tag{4.10}$$

Now assuming that M satisfies (4.10), consider an arbitrary input $f \in \mathcal{L}_2[0,\infty)$ and note that, if $f \in \mathcal{L}_2[0,\infty)$, then $f_T \in \mathcal{L}_2[0,\infty)$ for all $T \in [0,\infty)$. Since $f \in \mathcal{L}_2[0,\infty)$

and $M, \Gamma, \mathrm{B} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$, we can take limits as $T \to \infty$ to obtain (4.5).

A consequence of a system having a strict "mixed" small gain and passivity property, as defined in Definition 10, is that the system is guaranteed to have finite gain.

Lemma 2. *A system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ with a strict "mixed" small gain and passivity property (in the sense of Definition 10) has finite gain.*

Proof. Inequality (4.9) can be rewritten as

$$\begin{aligned}&\langle (Mf)_T, \Gamma^\sim\Gamma(Mf)_T\rangle + k\langle (Mf)_T, \mathrm{B}^\sim\mathrm{B}(Mf)_T\rangle \\ &\le \epsilon\langle f_T, \Gamma^\sim\Gamma f_T\rangle - l\langle f_T, \mathrm{B}^\sim\mathrm{B} f_T\rangle + 2\langle (Mf)_T, \mathrm{B}^\sim\mathrm{B} f_T\rangle + \eta \\ &\le \epsilon\langle f_T, \Gamma^\sim\Gamma f_T\rangle + l\langle f_T, \mathrm{B}^\sim\mathrm{B} f_T\rangle + 2\langle (Mf)_T, \mathrm{B}^\sim\mathrm{B} f_T\rangle + \eta.\end{aligned} \tag{4.11}$$

Let $\phi = \min\{1, k\}$, so that the first term of the above inequality is greater than or equal to

$$\begin{aligned}&\phi\left(\langle (Mf)_T, \Gamma^\sim\Gamma(Mf)_T\rangle + \langle (Mf)_T, \mathrm{B}^\sim\mathrm{B}(Mf)_T\rangle\right) \\ &= \phi\langle (Mf)_T, (\Gamma^\sim\Gamma + \mathrm{B}^\sim\mathrm{B})(Mf)_T\rangle \\ &= \phi\langle (Mf)_T, (Mf)_T\rangle\end{aligned}$$

using (4.8). That is, the first term of inequality (4.11) is greater than or equal to $\phi\|(Mf)_T\|^2$.

Let $\psi = \max\{\epsilon, l\}$, so that the last term of inequality (4.11) is less than or equal to

$$\begin{aligned}&\psi\left(\langle f_T, \Gamma^\sim\Gamma f_T\rangle + \langle f_T, \mathrm{B}^\sim\mathrm{B}) f_T\rangle\right) + 2\langle (Mf)_T, \mathrm{B}^\sim\mathrm{B} f_T\rangle + \eta \\ &= \psi\langle f_T, (\Gamma^\sim\Gamma + \mathrm{B}^\sim\mathrm{B}) f_T\rangle + 2\langle (Mf)_T, \mathrm{B}^\sim\mathrm{B} f_T\rangle + \eta \\ &= \psi\|f_T\|^2 + 2\langle (Mf)_T, \mathrm{B}^\sim\mathrm{B} f_T\rangle + \eta \text{ (using (4.8))} \\ &\le \psi\|f_T\|^2 + 2\|(Mf)_T\|\|\mathrm{B}^\sim\mathrm{B}\|\|f_T\| + \eta \text{ (using the Cauchy-Schwarz and} \\ &\quad \text{submultiplicative inequalities)} \\ &\le \psi\|f_T\|^2 + 2\|(Mf)_T\|\|f_T\| + \eta \text{ (since } \|\mathrm{B}^\sim\mathrm{B}\| \le 1).\end{aligned}$$

Since $\phi > 0$ we can conclude that

$$\|(Mf)_T\|^2 \le 2\bar{\phi}\|f_T\|\|(Mf)_T\| + \bar{\phi}\left(\psi\|f_T\|^2 + \eta\right)$$

where $\bar{\phi} := \frac{1}{\phi}$; and so

$$\begin{aligned}\|(Mf)_T\| &\le \bar{\phi}\|f_T\| + \sqrt{\bar{\phi}^2\|f_T\|^2 + \bar{\phi}\left(\psi\|f_T\|^2 + \eta\right)} \\ &\le \bar{\phi}\|f_T\| + \sqrt{\bar{\phi}^2\|f_T\|^2 + \bar{\phi}\psi\|f_T\|^2} + \sqrt{\bar{\phi}\eta}\end{aligned}$$

$$= \left(\bar{\phi} + \sqrt{\bar{\phi}^2 + \bar{\phi}\psi}\right) \|f_T\| + \sqrt{\bar{\phi}\eta}.$$

□

4.4 Feedback Interconnection Stability Result

An input-output stability result for the feedback interconnection shown in Fig. 4.1 is now provided. The result states that, if systems M_1 and M_2 each have associated with them a "mixed" small gain and passivity property, and furthermore, if the "mixed" small gain and passivity property associated with M_2 is strict, then the feedback interconnection is stable.[2]

Theorem 4. *Consider a feedback interconnection as shown in Fig. 4.1 and described by the equations*

$$e_1 = u_1 - M_2 e_2 \tag{4.12}$$
$$e_2 = u_2 + M_1 e_1 \tag{4.13}$$

where M_1 and M_2 causally map $\mathcal{L}_{2e}$ into $\mathcal{L}_{2e}$. Assume that for any u_1 and u_2 in $\mathcal{L}_2[0,\infty)$, there are solutions e_1 and e_2 in $\mathcal{L}_{2e}$. Suppose that there exist constants $\epsilon_1, k_1, l_1, \eta_1, \epsilon_2, k_2, l_2$ and η_2 such that

$$\begin{aligned}&- \langle \Gamma(M_1 f)_T, \Gamma(M_1 f)_T\rangle + \epsilon_1 \langle \Gamma f_T, \Gamma f_T\rangle - k_1 \langle B(M_1 f)_T, B(M_1 f)_T\rangle \\ &+ 2\langle B(M_1 f)_T, B f_T\rangle - l_1 \langle B f_T, B f_T\rangle + \eta_1 \geq 0\end{aligned} \tag{4.14}$$
$$\begin{aligned}&- \langle \Gamma(M_2 f)_T, \Gamma(M_2 f)_T\rangle + \epsilon_2 \langle \Gamma f_T, \Gamma f_T\rangle - k_2 \langle B(M_2 f)_T, B(M_2 f)_T\rangle \\ &+ 2\langle B(M_2 f)_T, B f_T\rangle - l_2 \langle B f_T, B f_T\rangle + \eta_2 \geq 0\end{aligned} \tag{4.15}$$

$\forall f \in \mathcal{L}_{2e}$, $\forall T \in [0,\infty)$, where Γ and B are as defined in Definition 10. Under these conditions, if

$$\begin{array}{cc} 0 \leq \epsilon_1 \leq 1 & l_1 + k_2 \geq 0 \\ 0 \leq \epsilon_2 < 1 & l_2 + k_1 > 0 \\ \multicolumn{2}{c}{k_2 > 0,\ l_2 > 0} \end{array}$$

then $u_1, u_2 \in \mathcal{L}_2[0,\infty)$ imply that $e_1, e_2, M_1 e_1, M_2 e_2 \in \mathcal{L}_2[0,\infty)$.

[2] In fact (corresponding to the choice of constants k_1 and l_1 below), the result permits M_1 to not have a "mixed" small gain and passivity property associated with it, provided that this "lack" of the property is compensated by the "strength" of the "mixed" small gain and passivity property associated with M_2. The constants defined in Theorem 4 and the conditions associated with them quantify these ideas of "lack", "strength" and compensation.

To avoid confusion, note that Γ as it appears in (4.14) and (4.15) is the same operator. Similarly, B as it appears in (4.14) and (4.15) is the same operator.[3] Thus M_1 and M_2 satisfy the same "mixed" small gain and passivity condition as far as frequency dependency is concerned; the constants ϵ_i, k_i, l_i and η_i may differ for $i = 1, 2$. Also note that with appropriate choices of Γ and B, Theorem 4 reduces to the passivity theorem ($\Gamma = \mathbf{0}$) and the small gain theorem (B$= \mathbf{0}$), respectively.

The input and output signal space for the feedback interconnection shown in Fig. 4.1 is the product space $\mathcal{L}_{2e} \times \mathcal{L}_{2e}$, and the elements of the input and output signal space are $u := \binom{u_1}{u_2}$ and $y := \binom{y_1}{y_2}$, respectively. Inner products in these spaces are derived by summing inner products in the component spaces. We proceed with a proof for Theorem 4 by summing the inner products of (4.14) and (4.15) to derive an inner product inequality describing the feedback interconnection. (This is as opposed to having two separate inequalities, namely (4.14) and (4.15), describing the feedback interconnection's component systems, namely M_1 and M_2, respectively.) Then appropriate manipulations of the inner product inequality give the desired stability result.

Proof. Truncating (4.12) and (4.13) gives

$$e_{1T} = u_{1T} - (M_2 e_2)_T \tag{4.16}$$
$$e_{2T} = u_{2T} + (M_1 e_1)_T. \tag{4.17}$$

For any $u_1, u_2 \in \mathcal{L}_2[0, \infty)$, for any $T \in [0, \infty)$,

$$\begin{aligned}
&- \langle \Gamma(M_1e_1)_T, \Gamma(M_1e_1)_T\rangle + \epsilon_1\langle \Gamma e_{1T}, \Gamma e_{1T}\rangle - k_1\langle \mathrm{B}(M_1e_1)_T, \mathrm{B}(M_1e_1)_T\rangle \\
&+ 2\langle \mathrm{B}(M_1e_1)_T, \mathrm{B}e_{1T}\rangle - l_1\langle \mathrm{B}e_{1T}, \mathrm{B}e_{1T}\rangle + \eta_1 - \langle \Gamma(M_2e_2)_T, \Gamma(M_2e_2)_T\rangle \\
&+ \epsilon_2\langle \Gamma e_{2T}, \Gamma e_{2T}\rangle - k_2\langle \mathrm{B}(M_2e_2)_T, \mathrm{B}(M_2e_2)_T\rangle + 2\langle \mathrm{B}(M_2e_2)_T, \mathrm{B}e_{2T}\rangle \\
&- l_2\langle \mathrm{B}e_{2T}, \mathrm{B}e_{2T}\rangle + \eta_2 \\
=&- \langle \Gamma e_{2T} - \Gamma u_{2T}, \Gamma e_{2T} - \Gamma u_{2T}\rangle + \epsilon_1\langle \Gamma e_{1T}, \Gamma e_{1T}\rangle - l_1\langle \mathrm{B}e_{1T}, \mathrm{B}e_{1T}\rangle \\
&+ 2\langle \mathrm{B}e_{2T} - \mathrm{B}u_{2T}, \mathrm{B}e_{1T}\rangle - k_1\langle \mathrm{B}e_{2T} - \mathrm{B}u_{2T}, \mathrm{B}e_{2T} - \mathrm{B}u_{2T}\rangle + \eta_1 \\
&- \langle \Gamma u_{1T} - \Gamma e_{1T}, \Gamma u_{1T} - \Gamma e_{1T}\rangle + \epsilon_2\langle \Gamma e_{2T}, \Gamma e_{2T}\rangle - l_2\langle \mathrm{B}e_{2T}, \mathrm{B}e_{2T}\rangle \\
&+ 2\langle \mathrm{B}u_{1T} - \mathrm{B}e_{1T}, \mathrm{B}e_{2T}\rangle - k_2\langle \mathrm{B}u_{1T} - \mathrm{B}e_{1T}, \mathrm{B}u_{1T} - \mathrm{B}e_{1T}\rangle + \eta_2 \\
=&- \langle e_{1T}, [(1-\epsilon_1)\Gamma^\sim\Gamma + (l_1 + k_2)\mathrm{B}^\sim\mathrm{B}]\, e_{1T}\rangle - \langle u_{1T}, (\Gamma^\sim\Gamma + k_2\mathrm{B}^\sim\mathrm{B})u_{1T}\rangle \\
&- \langle e_{2T}, [(1-\epsilon_2)\Gamma^\sim\Gamma + (l_2 + k_1)\mathrm{B}^\sim\mathrm{B}]\, e_{2T}\rangle - \langle u_{2T}, (\Gamma^\sim\Gamma + k_1\mathrm{B}^\sim\mathrm{B})u_{2T}\rangle \\
&+ 2\langle e_{1T}, (\Gamma^\sim\Gamma + k_2\mathrm{B}^\sim\mathrm{B})u_{1T}\rangle + 2\langle e_{2T}, (\Gamma^\sim\Gamma + k_1\mathrm{B}^\sim\mathrm{B})u_{2T}\rangle \\
&- 2\langle e_{1T}, \mathrm{B}^\sim\mathrm{B}u_{2T}\rangle + 2\langle e_{2T}, \mathrm{B}^\sim\mathrm{B}u_{1T}\rangle + \eta_1 + \eta_2
\end{aligned}$$

[3] In the LTI case, this relates to the requirement that a common frequency interval can be found on which both systems in the feedback interconnection are "input and output strictly passive and have gain less than one" (see Chapter 3 for details).

using (4.16) and (4.17) to substitute in for $(M_2e_2)_T$ and $(M_1e_1)_T$, respectively, and then rearranging. Using (4.14) and (4.15), the first and thus the last member of this equality is greater than or equal to zero. In this inequality, set $\bar{\eta} := \eta_1 + \eta_2$ for convenience. In other words, for any $u_1, u_2 \in \mathcal{L}_2[0,\infty)$ and any $T \in [0,\infty)$, we know that there exist constants $\epsilon_1, k_1, l_1, \epsilon_2, k_2, l_2$ and $\bar{\eta}$ such that

$$\begin{aligned}
&\langle e_{1T}, [(1-\epsilon_1)\Gamma^{\sim}\Gamma + (l_1+k_2)\mathrm{B}^{\sim}\mathrm{B}]\, e_{1T}\rangle + \langle e_{2T}, [(1-\epsilon_2)\Gamma^{\sim}\Gamma + (l_2+k_1)\mathrm{B}^{\sim}\mathrm{B}]\, e_{2T}\rangle \\
&\leq 2\langle e_{1T}, (\Gamma^{\sim}\Gamma + k_2\mathrm{B}^{\sim}\mathrm{B})u_{1T}\rangle + 2\langle e_{2T}, (\Gamma^{\sim}\Gamma + k_1\mathrm{B}^{\sim}\mathrm{B})u_{2T}\rangle - 2\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle + \\
&2\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle - \langle u_{1T}, (\Gamma^{\sim}\Gamma + k_2\mathrm{B}^{\sim}\mathrm{B})u_{1T}\rangle - \langle u_{2T}, (\Gamma^{\sim}\Gamma + k_1\mathrm{B}^{\sim}\mathrm{B})u_{2T}\rangle + \bar{\eta}
\end{aligned} \tag{4.18}$$

$\forall e_1, e_2 \in \mathcal{L}_{2e}$, $\forall T \in [0,\infty)$.

The LHS of inequality (4.18) is equal to

$$\begin{aligned}
&(1-\epsilon_1)\langle e_{1T}, \Gamma^{\sim}\Gamma e_{1T}\rangle + (l_1+k_2)\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}e_{1T}\rangle + (1-\epsilon_2)\langle e_{2T}, \Gamma^{\sim}\Gamma e_{2T}\rangle \\
&+ (l_2+k_1)\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}e_{2T}\rangle,
\end{aligned}$$

which is greater than or equal to

$$(1-\epsilon_2)\langle e_{2T}, \Gamma^{\sim}\Gamma e_{2T}\rangle + (l_2+k_1)\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}e_{2T}\rangle \tag{4.19}$$

since $1-\epsilon_1, l_1+k_2 \geq 0$. Let $\sigma = \min\{1-\epsilon_2, l_2+k_1\}$ (noting that $\sigma > 0$) such that the term denoted by (4.19) is greater than or equal to

$$\begin{aligned}
\sigma\left(\langle e_{2T}, \Gamma^{\sim}\Gamma e_{2T}\rangle + \langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}e_{2T}\rangle\right) &= \sigma\langle e_{2T}, (\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B})e_{2T}\rangle \\
&= \sigma\|e_{2T}\|^2
\end{aligned}$$

using (4.8).

The RHS of inequality (4.18) is equal to

$$\begin{aligned}
&2\langle e_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle + 2k_2\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle + 2\langle e_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle + 2k_1\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle \\
&- 2\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle + 2\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle - \langle u_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle - k_2\langle u_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle \\
&- \langle u_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle - k_1\langle u_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle + \bar{\eta}
\end{aligned}$$

which is less than or equal to

$$\begin{aligned}
&2|\langle e_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle| + 2k_2|\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle| + 2|\langle e_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle| + 2|k_1||\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle| \\
&+ 2|\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle| + 2|\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle| + \langle u_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle + k_2\langle u_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle \\
&+ \langle u_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle + |k_1|\langle u_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle + \bar{\eta}.
\end{aligned} \tag{4.20}$$

Let $\rho = \max\{1, k_2, |k_1|\}$ such that the term denoted by (4.20) is less than or equal to

$$2\rho(|\langle e_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle| + |\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle| + |\langle e_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle| + |\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle|) + 2(|\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle| + |\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle|) + \rho(\langle u_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle + \langle u_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle + \langle u_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle + \langle u_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle) + \bar{\eta}. \tag{4.21}$$

Note that

$$\begin{aligned}&\rho(\langle u_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle + \langle u_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle + \langle u_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle + \langle u_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle)\\ &= \rho(\langle u_{1T}, (\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B})u_{1T}\rangle + \langle u_{2T}, (\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B})u_{2T}\rangle)\\ &= \rho(\|u_{1T}\|^2 + \|u_{2T}\|^2)\end{aligned}$$

using (4.8). So the term denoted by (4.21) is equal to

$$2\rho(|\langle e_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle| + |\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle| + |\langle e_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle| + |\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle|) + 2(|\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle| + |\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle|) + \rho(\|u_{1T}\|^2 + \|u_{2T}\|^2) + \bar{\eta}$$

which is less than or equal to

$$2\rho(\|\Gamma^{\sim}\Gamma\| + \|\mathrm{B}^{\sim}\mathrm{B}\|)(\|e_{1T}\|\|u_{1T}\| + \|e_{2T}\|\|u_{2T}\|) + 2\|\mathrm{B}^{\sim}\mathrm{B}\|(\|e_{1T}\|\|u_{2T}\| + \|e_{2T}\|\|u_{1T}\|) + \rho(\|u_{1T}\|^2 + \|u_{2T}\|^2) + \bar{\eta} \tag{4.22}$$

by using the Cauchy-Schwarz and submultiplicative inequalities. Eliminating e_{1T} using (4.16), using the triangle inequality, and then applying Lemma 2 to M_2 shows that the term denoted by (4.22) is less than or equal to

$$2\rho(\|\Gamma^{\sim}\Gamma\| + \|\mathrm{B}^{\sim}\mathrm{B}\|)((\|u_{1T}\| + \kappa\|e_{2T}\| + \xi)\|u_{1T}\| + \|e_{2T}\|\|u_{2T}\|) + \bar{\eta} + 2\|\mathrm{B}^{\sim}\mathrm{B}\|((\|u_{1T}\| + \kappa\|e_{2T}\| + \xi)\|u_{2T}\| + \|e_{2T}\|\|u_{1T}\|) + \rho(\|u_{1T}\|^2 + \|u_{2T}\|^2)$$

where the non-negative constants κ and ξ exist due to the boundedness of M_2. Since $\sigma > 0$, we can conclude that

$$\|e_{2T}\|^2 \leq 2\bar{b}(T)\|e_{2T}\| + \bar{c}(T), \tag{4.23}$$

where $\bar{b}(T)$ and $\bar{c}(T)$ tend to finite values $\bar{b}$ and $\bar{c}$, respectively, as $T \to \infty$, since $u_1, u_2 \in \mathcal{L}_2[0,\infty)$. From (4.23)

$$\|e_{2T}\| \leq \bar{b}(T) + (\bar{b}(T)^2 + \bar{c}(T))^{\frac{1}{2}}$$

$\forall T \in [0,\infty)$, and remains bounded as $T \to \infty$. So $e_2 \in \mathcal{L}_2[0,\infty)$. From Lemma 2 the same holds for M_2e_2, ie: $M_2e_2 \in \mathcal{L}_2[0,\infty)$. By (4.12) and (4.13) it follows that $e_1, M_1e_1 \in \mathcal{L}_2[0,\infty)$. □

4.5 Conclusions

An input-output stability result was obtained for a standard feedback interconnection containing two causal, nonlinear systems, where each system had a "mixed" small gain and passivity assumption associated with it. It was indicated that the "mixed" small gain and passivity property reduced to a description of a system that was input and output strictly passive; or alternatively, to a description of a system that had gain less than one, when certain operators were appropriately defined. The "mixed" small gain and passivity property also captured a notion of "blending" of the small gain and passivity ideas, and thus described a class of systems that was larger than the class of passive systems together with the class of systems with small gain.

5. A SCALED LTI ν-GAP METRIC FRAMEWORK FOR THE STRUCTURED LTV UNCERTAINTY PROBLEM[1]

5.1 Introduction

A typical stability robustness problem is of the following nature. Suppose that a LTI modelled (or nominal) plant P_0 is connected to a LTI controller K, as shown in Fig. 5.1, such that the closed-loop system is internally stable. That is, the closed-loop system is well-posed, and each of the four transfer function matrices mapping the signals v_1 and v_2, to y and u, are stable [104, Lemma 5.3]. The question then asked is whether the controller K will successfully stabilize the system if P_0 is subject to structured LTV uncertainty Δ, as shown in Fig. 5.2, where the transfer function matrix $F(s)$ is constructed by relating P_0 and Δ via an upper linear fractional transformation (LFT).

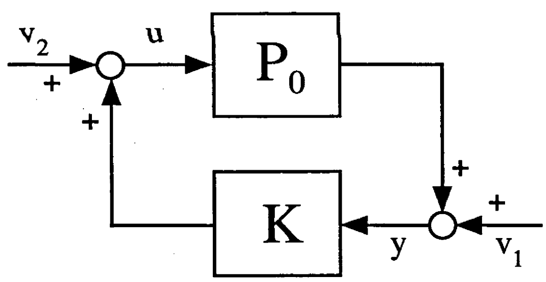

Fig. 5.1: Nominal closed-loop system.

Traditionally, a scaled small gain condition has been one of the tools used as a means for determining stability robustness in regards to this problem [17, 20, 80]. Provided that the nominal plant and controller are LTI, and the uncertainties are of a structured form, then this scaled small gain condition is necessary and sufficient [20, 80]. A detailed description of the scaled small gain condition is provided

[1] Content from this chapter was published in [37]; and will be submitted to a journal in the immediate future.

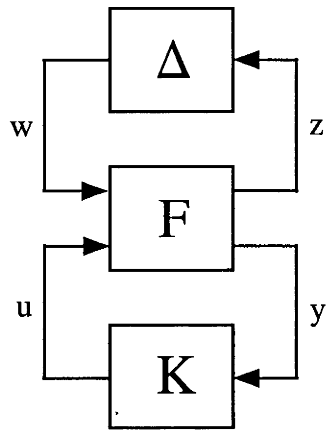

Fig. 5.2: Robust stability problem.

later.

In this chapter, the scaled small gain condition for determining system stability robustness to structured LTV uncertainties, is represented in a scaled LTI ν-gap metric framework. An advantage of doing this lies in the fact that, while both conditions can be checked by solving a LMI feasibility problem, the scaled LTI ν-gap metric condition in this regard is independent of the controller. That is, the scaled LTI ν-gap metric condition involves knowledge of K, to the extent that the condition requires a scaled LTI ν-gap quantity (independent of K) be computed, and compared to a generalized robust stability margin (dependent on K). Unsurprisingly, the scaled LTI ν-gap metric condition may provide a more conservative test for stability robustness than the scaled small gain condition (since the less knowledge one utilizes regarding the controller, the more careful one has to be). However, the LTI ν-gap metric is the least conservative metric (of the gap metrics) in the sense that the existence of at least one destabilizing controller is guaranteed if the distance between plants (as measured by the ν-gap metric) is "large".

Indeed, the ν-gap metric [90] is an invaluable tool used frequently in the study of stability robustness. As mentioned, it provides a measure of difference or "distance" between two systems from a feedback perspective; thus a controller that stabilizes one system will also stabilize the other provided that the distance between the two systems (as measured by the ν-gap metric) is small (see part (i) of Lemma 1). On the other hand, there will always exist at least one controller that will stabilize one system but destabilize the other if the distance between the two systems is large. Time-varying ν-gap metrics, as opposed to LTI ν-gap metrics, are not generally analytically computable. For example, if we consider the calculation of the time-varying ν-gap metric defined by [3, Definition III.4], between a SISO LTI plant $P_0 = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right)$ and the LTV output-multiplicatively perturbed plant $P_1 = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1 + \epsilon \sin(at+b) & 0 \end{array}\right)$ where $\epsilon \in [0,1]$, we see that, using [3, Definition III.4], one is required to solve generalized differential Riccati equations to obtain time-varying normalized graph symbols. This is not analytically possible, even for the basic example mentioned.

A description of the scaled small gain condition is provided in Section 5.2. Use of the scaled small gain condition for determining system stability robustness is extended to the scaled LTI ν-gap metric framework in Section 5.3. To mathematically evaluate the scaled LTI ν-gap metric stability condition, a LMI feasibility problem is solved. The theoretical construction of this LMI feasibility problem is described in Section 5.4, and a complete solution algorithm is provided in Section 5.5. An example of the implementation of the solution algorithm is provided in Section 5.6. Section 5.7 concludes the chapter.

Preliminaries The following is a specific account of the mathematical notation used throughout this chapter. (Note that some of the symbols used here may have different meanings to what they had in previous chapters.) The space $\mathcal{L}_2(-\infty,\infty)$ is a space consisting of Lebesgue measurable functions with finite norm. $\mathcal{L}_2[0,\infty)$ is the subspace of $\mathcal{L}_2(-\infty,\infty)$ with functions zero for $t < 0$. $\mathcal{R}$ denotes the set of proper real rational transfer function matrices. $\mathcal{L}_\infty(j\mathbb{R})$ is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on $j\mathbb{R}$. The Hardy space $\mathcal{H}_\infty$ is the closed subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane (RHP), with norm denoted $\|\cdot\|_\infty$. In other words, $\mathcal{H}_\infty$ is the space of transfer functions of stable, LTI, continuous-time systems. $\mathcal{RH}_\infty$ denotes the subspace of $\mathcal{H}_\infty$ where transfer function matrices are proper and real rational. The $\mathcal{L}_2$-induced norm for LTV operators will be denoted by $\|\cdot\|$. For LTI systems, the $\mathcal{L}_2$-induced norm is precisely equal to $\|\cdot\|_\infty$.

For a general matrix $X = [x_{ij}] \in \mathbb{C}^{r \times s}$, X^* denotes the complex conjugate transpose $[\bar{x}_{ji}]$. For a transfer function matrix $X(s) \in \mathcal{R}^{r \times s}$, $X^\sim(s)$ is defined to mean $X(-s)^T$; while $X(j\omega)^*$ denotes the complex conjugate transpose of the frequency response function $X(j\omega)$ at each frequency, ie: $X(j\omega)^* = X(-j\omega)^T$. Let $X \in \mathbb{C}^{(r_1+r_2)\times(s_1+s_2)}$ be partitioned as follows:

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

and let $Y_l \in \mathbb{C}^{s_2 \times r_2}$ and $Y_u \in \mathbb{C}^{s_1 \times r_1}$. The notation

$$F_l(X, Y_l) := X_{11} + X_{12} Y_l (I - X_{22} Y_l)^{-1} X_{21}$$

and

$$F_u(X, Y_u) := X_{22} + X_{21}Y_u(I - X_{11}Y_u)^{-1}X_{12}$$

refers to the standard lower and upper linear fractional representations, respectively, as shown in Fig. 5.3. $X_1 \star X_2$ denotes the interconnection of two LFTs known as the Redheffer star-product, as shown in Fig. 5.4. The notation

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

refers to a system realization (A, B, C, D). Well-posedness of closed-loops is assumed throughout the chapter.

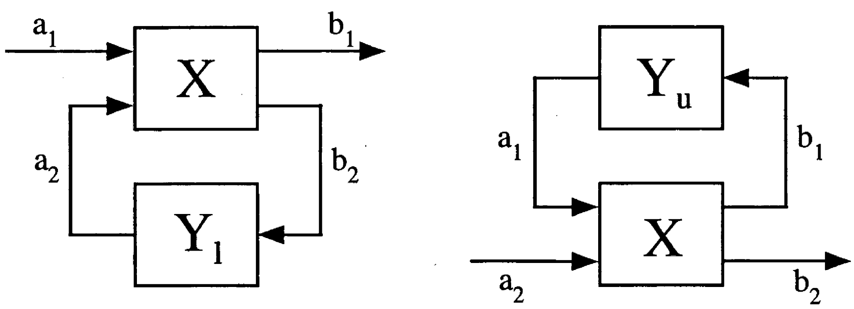

Fig. 5.3: Lower and upper LFTs.

5.2 The Scaled Small Gain Condition

A description of the scaled small gain condition that is of interest to this chapter is provided as follows [20]. Consider the uncertain system shown in Fig. 5.2. Suppose that the generalized system F is partitioned as

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \tag{5.1}$$

where $F_{11} \in \mathcal{R}^{p\times q}$, $F_{12} \in \mathcal{R}^{p\times m}$, $F_{21} \in \mathcal{R}^{n\times q}$ and $F_{22} \in \mathcal{R}^{n\times m}$. Furthermore, let a stabilizable and detectable realization for $F \in \mathcal{R}^{(p+n)\times(q+m)}$ be given by

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right). \tag{5.2}$$

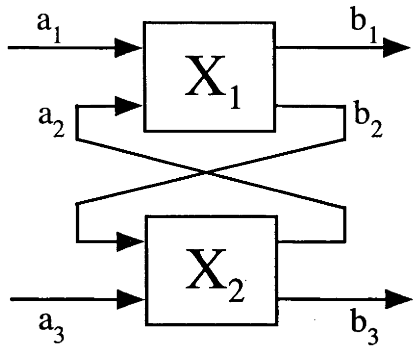

Fig. 5.4: Redheffer star product.

The nominal plant is denoted $P_0 := F_u(F, 0) = F_{22}$; and the controller is denoted $K \in \mathcal{R}^{m \times n}$. Let the interconnection of P_0 and K, as shown in Fig. 5.1, be denoted by $[P_0, K]$. This interconnection is said to be internally stable if it is well-posed and each of the four transfer functions mapping the signals v_1 and v_2, to y and u, are stable; that is, they belong to RH_∞ [104]. Recall that it is our intention to assume that this interconnection is internally stable. Suppose that P_0 has an inherited realization (A, B_2, C_2) that is also stabilizable and detectable. Then $[P_0, K]$ is internally stable if and only if the system in Fig. 5.5 is internally stable [104, Lemma 12.2]. Denote $Z := F_l(F, K)$. The system in Fig. 5.2 may be reduced to the system shown in Fig. 5.6.

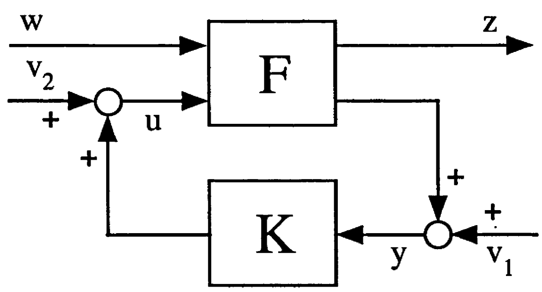

Fig. 5.5: Internal stability of $F_l(F, K)$.

Let us define a block-diagonal uncertainty set

$$\mathbf{\Delta} := \{\Delta = \text{diag}(\Delta_1 \ldots \Delta_k) : \Delta_i \text{ is a } q_i \times p_i \text{ causal LTV operator and } \|\Delta\| \leq 1\},$$

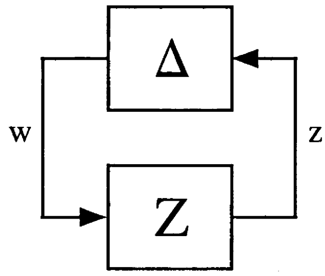

Fig. 5.6: Reduced stability robustness problem.

where $q := q_1 + \ldots + q_k$ and $p := p_1 + \ldots + p_k$. Here we make the observation that, since $\boldsymbol{\Delta}$ is a subset of the unit ball of causal LTV operators, then clearly the standard small gain condition

$$\|Z\|_\infty < 1$$

for unstructured uncertainty is a sufficient test for input-output stability of the system shown in Fig. 5.6 [20]. Now, associate with $\boldsymbol{\Delta}$ a set of scalings that commute with the set of perturbations. In particular, we will choose a set of constant diagonal matrix pairs which share the same scalar coefficients, denoted

$$\begin{aligned} \boldsymbol{D} :=& \{(D_l, D_r) : D_l = \mathrm{diag}(d_1 I_{q_1}, \ldots, d_k I_{q_k}), D_r = \mathrm{diag}(d_1 I_{p_1}, \ldots, d_k I_{p_k}), d_i \in \mathbb{R}, \\ & d_i > 0\} \end{aligned}$$

such that

$$\Delta = D_l \Delta D_r^{-1}$$

$\forall \Delta \in \boldsymbol{\Delta}$. The stability of the system in Fig. 5.7 is equivalent to the stability of the system in Fig. 5.6 (since it is the same system). This means that if we can find an element $\widetilde{D} = (D_l, D_r)$ of $\boldsymbol{D}$ satisfying

$$\|D_r Z D_l^{-1}\|_\infty < 1$$

then we can guarantee that the system in Fig. 5.6 is input-output stable. Since the identity matrices (of suitable dimensions) are members of $\boldsymbol{D}$, there always exists a $\widetilde{D} = (D_l, D_r) \in \boldsymbol{D}$ such that $\|D_r Z D_l^{-1}\|_\infty \leq \|Z\|_\infty$. Thus, the scaled small gain condition provides a less conservative test for stability then the standard small gain condition [20]. In fact, it is furthermore possible to obtain the following result [20].

Theorem 5. *[20, Theorem 9.6] Consider a causal, LTI system with transfer function matrix $Z \in \mathcal{RH}_\infty$. The following are equivalent:*

(i) The system in Fig. 5.6 is stable for all $\Delta \in \boldsymbol{\Delta}$.

(ii) The inequality $\inf_{\tilde{D}=(D_l,D_r)\in \boldsymbol{D}} \|D_r Z D_l^{-1}\|_\infty < 1$ *holds.*

A proof of Theorem 5 is provided in [20]. A discrete-time version of Theorem 5 can be found in [17, 80]. Note there exist other cases for which the scaled small gain condition is a necessary and sufficient condition for stability. Another case that will be utilized later in this chapter is where the uncertainties are LTI and structured such that they consist of two diagonal blocks. Reference [104, Chapter 11] provides a thorough discussion of this case. In the following section, the scaled small gain condition in Theorem 5 is extended into a scaled LTI ν-gap metric framework.

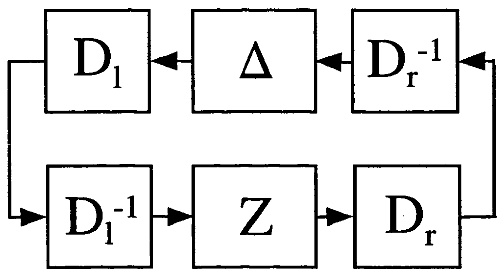

Fig. 5.7: Scaled robust stability problem.

5.3 The Scaled LTI ν-gap Metric Condition

We now present a result where the scaled small gain condition for determining stability robustness of the system shown in Fig. 5.2, is placed into a scaled LTI ν-gap metric framework. The following is claimed: given that a controller K stabilizes a nominal plant P_0 with generalized robust stability margin $b_{P_0,K}$, if a certain LTI quantity is smaller than $b_{P_0,K}$, then the same controller will stabilize the system when subjected to structured LTV uncertainty. If the LTI quantity is equal to or larger than $b_{P_0,K}$, then the controller may or may not stabilize the system when subject to uncertainty.

Let $\boldsymbol{\delta} := \{\delta : \delta \in \mathcal{RH}_\infty^{q\times p}, \|\delta\|_\infty \le 1\}$ and $\boldsymbol{\delta_o} := \{\delta : \delta \in \mathcal{RH}_\infty^{q\times p}, \|\delta\|_\infty < 1\}$ denote full-block sets of LTI uncertainties. The generalized robust stability margin, $b_{P_0,K}$; the optimal generalized robust stability margin, $b_{opt}(P_0) := \sup_K b_{P_0,K}$; and the LTI ν-gap metric, denoted $\delta_\nu(P_0, P_1)$ where $P_0 \in \mathcal{R}^{n\times m}$ and $P_1 \in \mathcal{R}^{n\times m}$, are defined as in [86] and Chapter 2.

Theorem 6. *Let a generalized plant $F \in \mathcal{R}^{(p+n)\times(q+m)}$ be partitioned as in* (5.1) *and have a stabilizable and detectable realization as given by* (5.2)*. Let $P_0 := F_u(F, 0)$ be the nominal plant with an inherited realization (A, B_2, C_2) which is stabilizable and detectable*[2]*; and let $K \in \mathcal{R}^{m\times n}$ be a stabilizing controller for P_0, with a given stabilizable and detectable realization. Consider the uncertainty sets $\boldsymbol{\Delta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\delta_o}$ and the set of constant diagonal matrix pairs $\boldsymbol{D}$ as defined above. Suppose that each $\Delta \in \boldsymbol{\Delta}$ and each $\delta \in \boldsymbol{\delta}$ has a given stabilizable and detectable realization; and that each induced realization for $F_u(F, \Delta)$ and $F_u(F, D_l^{-1}\delta D_r)$ is stabilizable and detectable (as defined in Appendix A). If*

$$\inf_{\widetilde{D}=(D_l,D_r)\in\boldsymbol{D}} \sup_{\delta\in\boldsymbol{\delta_o}} \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r)) < b_{P_0,K}, \tag{5.3}$$

then $[P_{LTV}, K]$ is internally stable for all $\Delta \in \boldsymbol{\Delta}$, where $P_{LTV} := F_u(F, \Delta)$.

Proof. We have

$$\begin{aligned}
&\inf_{\widetilde{D}=(D_l,D_r)\in\boldsymbol{D}} \sup_{\delta\in\boldsymbol{\delta_o}} \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r)) < b_{P_0,K} \\
&\Leftrightarrow \exists \widetilde{D} \in \boldsymbol{D} : \forall \delta \in \boldsymbol{\delta}\ \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r)) < b_{P_0,K} && (5.4)\\
&\Rightarrow \exists \widetilde{D} \in \boldsymbol{D} : \forall \delta \in \boldsymbol{\delta} \text{ the system in Fig. 5.8 is internally stable (using the} \\
&\quad\ \text{stability result associated with the LTI } \nu\text{-gap metric that is provided} \\
&\quad\ \text{in Section 2.2.3 and in [86, 90])} && (5.5)\\
&\Leftrightarrow \exists \widetilde{D} \in \boldsymbol{D} : \|D_r F_l(F, K) D_l^{-1}\|_\infty < 1 \text{ (using the small gain theorem in} \\
&\quad\ \text{[104, Theorem 9.1], since } F_l(F, K) \in \mathcal{RH}_\infty \text{ as shown in Appendix A)} \\
&\Leftrightarrow \inf_{\widetilde{D}=(D_l,D_r)\in\boldsymbol{D}} \|D_r F_l(F, K) D_l^{-1}\|_\infty < 1 \\
&\Leftrightarrow \forall \Delta \in \boldsymbol{\Delta} \text{ the system in Fig. 5.2 is internally stable (from Theorem 5).}
\end{aligned}$$

□

The only implication in the above proof that is not necessary and sufficient is the one that relates (5.4) to (5.5), which is based on Lemma 1 (see [86, Remark 3.11]). Therefore, the scaled LTI ν-gap metric stability condition (5.3), even though only a sufficient condition, is still the strongest result one could derive in the following sense. The LTI ν-gap metric is strongly necessary in the sense that there always exists a controller K satisfying $\delta_\nu(\cdot,\cdot) \not< b_{P,K}$ for which internal stability is lost (see [90, Theorem 4.5] or [86, Remark 3.11] for further details), and hence there always exists a K that does not satisfy inequality (5.3) for which robust internal stability of the system shown in Fig. 5.8 is lost.

[2] Such an assumption is a standard assumption in $\mathcal{H}_\infty$ control and is necessary and sufficient for F to be internally stabilizable via a controller connecting y to u.

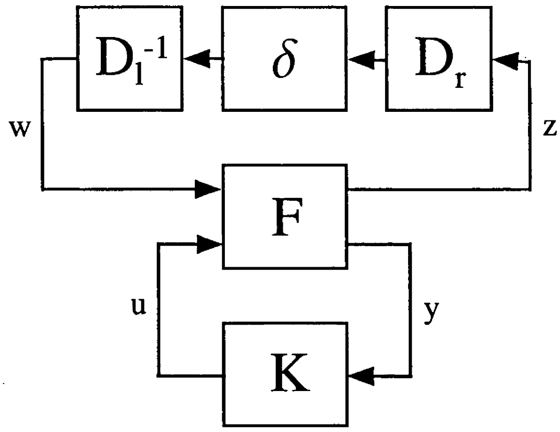

Fig. 5.8: Internal stability of LTI system.

An observation in regards to the scaled small gain condition when compared to the scaled LTI ν-gap metric condition is as follows. The scaled small gain condition: does there exist a $\widetilde{D} = (D_l, D_r) \in \boldsymbol{D}$ such that

$$\|D_r F_l(F, K) D_l^{-1}\|_\infty < 1,$$

is often checked directly using LMI techniques. Part (ii) of the following Lemma shows that the condition is equivalent to a convex condition over the positive scaling set $\boldsymbol{D}$; and by part (iii) of the Lemma, checking the condition reduces to solving a LMI feasibility problem [20]. (For simplicity, it is assumed in the Lemma below that $\widetilde{D} = D_r = D_l$.)

Lemma 3. *[20, Proposition 8.6] Suppose $Z \in \mathcal{RH}_\infty$ with transfer function matrix $Z(s) = C(sI - A)^{-1}B + D$, and A is Hurwitz of order n. Then the following are equivalent:*

(i) $\exists \widetilde{D} \in \boldsymbol{D} : \|\widetilde{D} Z \widetilde{D}^{-1}\|_\infty < 1$*;*

(ii) $\exists \widetilde{D} \in \boldsymbol{D} : Z^* \widetilde{D} Z - \widetilde{D} < 0$*;*

(iii) $\exists \widetilde{D} \in \boldsymbol{D}$ *and a symmetric $n \times n$ matrix $X > 0$ such that*

$$\begin{pmatrix} C^* \\ D^* \end{pmatrix} \widetilde{D} \begin{pmatrix} C & D \end{pmatrix} + \begin{pmatrix} A^*X + XA & XB \\ B^*X & -\widetilde{D} \end{pmatrix} < 0.$$

Refer to [20] for the proof. Note that checking the scaled small gain condition in this way is dependent on information regarding the controller K. That is, for each different K, one must solve a different LMI feasibility problem (for instance) to determine stability. However, the LHS of (5.3) is independent of K. In the subsequent sections, it will be shown that the LTI quantity on the LHS of (5.3) can also be computed by solving a LMI feasibility problem (in association with a bisectional search). Since the quantity on the LHS of (5.3) is independent of K, this means that the (entire solution algorithm involving the) associated LMI feasibility problem has only got to be solved once (given F); and is then compared to $b_{P_0,K}$ which is computed as required.

5.4 A LMI Feasibility Problem

It was stated above that the LTI quantity on the LHS of (5.3) can be computed via solving a LMI feasibility problem. This result is presented later in this section (see Theorem 7). First, some preliminary results are required.

The first result is a minor but important extension of [13, Proposition 1.1], and relates a LTI ν-gap metric and a transfer function matrix stability and small gain concept. First, a LTI system $\tilde{R}$ (dependent on some strictly proper LTI system P_1 and some number $\beta \in (0, b_{opt}(P_1)))$ is introduced as follows. Suppose that P_1 has a stabilizable and detectable realization $(A_{P_1}, B_{P_1}, C_{P_1})$. Let $X = X^* \geq 0$ be the stabilizing solution to the generalized control algebraic Riccati equation (GCARE)

$$A_{P_1}^* X + X A_{P_1} - X B_{P_1} B_{P_1}^* X + C_{P_1}^* C_{P_1} = 0$$

and $Z = Z^* \geq 0$ be the stabilizing solution to the generalized filtering algebraic Riccati equation (GFARE)

$$A_{P_1} Z + Z A_{P_1}^* - Z C_{P_1}^* C_{P_1} Z + B_{P_1} B_{P_1}^* = 0.$$

Let $\gamma := \frac{1}{\beta}$. Define, as per [12], a transfer function matrix $\tilde{R} \in \mathcal{R}^{(n+m)\times(m+n)}$ via the realization

$$\left(\begin{array}{c|cc} A_{\tilde{R}} & B_{\tilde{R}_1} & B_{\tilde{R}_2} \\ \hline C_{\tilde{R}_1} & 0 & I \\ C_{\tilde{R}_2} & \sqrt{\gamma^2-1}I & 0 \end{array}\right), \tag{5.6}$$

where $Y := \frac{1}{\gamma^2-1} Z$, $\bar{Y} := Y(I - XY)^{-1}$ and

$$\begin{aligned} A_{\tilde{R}} &:= A_{P_1} - B_{P_1} B_{P_1}^* X - \gamma^2 \bar{Y} C_{P_1}^* C_{P_1} \\ B_{\tilde{R}_1} &:= \frac{\gamma}{\sqrt{\gamma^2-1}} (I - YX)^{-1} B_{P_1} \\ B_{\tilde{R}_2} &:= \gamma \bar{Y} C_{P_1}^* \\ C_{\tilde{R}_1} &:= -\gamma C_{P_1} \\ C_{\tilde{R}_2} &:= -\gamma B_{P_1}^* X. \end{aligned}$$

Note it was shown in [14] that $\tilde{R}$ is invertible in $\mathcal{R}^{(m+n)\times(n+m)}$. In fact, $\tilde{R}$ in this dissertation is equivalent to R^{-1} in [12–14], where

$$R := \left(\begin{array}{c|cc} A_{P_1} + \frac{1}{\gamma^2-1}B_{P_1}B_{P_1}^*X + \frac{\gamma^2}{\gamma^2-1}\bar{Y}X^*B_{P_1}B_{P_1}^*X & B_{\tilde{R}_2} & \frac{\gamma}{\gamma^2-1}(I+\bar{Y}X^*)B_{P_1} \\ \hline -\frac{1}{\sqrt{\gamma^2-1}}C_{\tilde{R}_2} & 0 & \frac{1}{\sqrt{\gamma^2-1}}I \\ -C_{\tilde{R}_1} & I & 0 \end{array}\right).$$

It was also shown in [14] that $(R_{12})^{-1}, (R_{21})^{-1} \in \mathcal{RH}_\infty$; and it was noted in [89] that the realization for R was naturally induced by the problem structure described in [12–14,89].

The realization (5.6) is stabilizable and detectable since there exist matrices $F_{\tilde{R}} := \begin{pmatrix} 0 \\ -C_{\tilde{R}_1} \end{pmatrix}$ and $L_{\tilde{R}} := \begin{pmatrix} -B_{\tilde{R}_2} & 0 \end{pmatrix}$ such that $A_{\tilde{R}} + \begin{pmatrix} B_{\tilde{R}_1} & B_{\tilde{R}_2} \end{pmatrix} F_{\tilde{R}}$ and $A_{\tilde{R}} + L_{\tilde{R}} \begin{pmatrix} C_{\tilde{R}_1} \\ C_{\tilde{R}_2} \end{pmatrix}$, respectively, are Hurwitz.[3] The extension to [13, Proposition 1.1] is as follows.

Lemma 4. *Given two LTI systems P_1, $P_2 \in \mathcal{R}^{n\times m}$, where $P_1(s) \to 0$ as $s \to \infty$ but P_2 is not necessarily strictly proper, and a number $\beta \in (0, b_{opt}(P_1))$, there exists a LTI system $\tilde{R}$ (dependent on P_1 and β) such that*

$$\delta_\nu(P_1, P_2) \le \beta$$
$$\Updownarrow$$
$$||F_l(\tilde{R}, P_2)||_\infty \le 1 \text{ and the system in Fig. 5.9 is internally stable.} \tag{5.7}$$

Proof. The difference between the statement of Lemma 4 and the statement of [13, Proposition 1.1] occurs at (5.7). In [13, Proposition 1.1], this condition reads

$$||F_l(\tilde{R}, P_2)||_\infty \le 1 \text{ and } F_l(\tilde{R}, P_2) \in \mathcal{RH}_\infty;$$

whereas in Lemma 4, this condition reads

$$||F_l(\tilde{R}, P_2)||_\infty \le 1 \text{ and the system in Fig. 5.9 is internally stable.}$$

Hence the aim of the proof is to establish that $F_l(\tilde{R}, P_2) \in \mathcal{RH}_\infty$ if and only if the system in Fig. 5.9 is internally stable.

($\Leftarrow$) This way is obvious since internal stability of the system in Fig. 5.9 is equivalent to the transfer function matrix mapping $(f'\ e_1'\ e_2')'$ to $(g'\ a_1'\ a_2')'$ (as shown in Fig. 5.10) being in $\mathcal{RH}_\infty$. Since $F_l(\tilde{R}, P_2)$ maps f to g, $F_l(\tilde{R}, P_2)$ thus belongs to $\mathcal{RH}_\infty$.

[3] Note that $A_{\tilde{R}} - B_{\tilde{R}_2}C_{\tilde{R}_1} = A_{P_1} - B_{P_1}B_{P_1}^*X$.

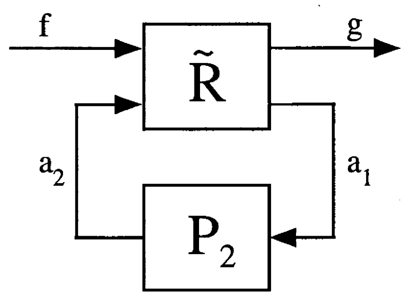

Fig. 5.9: Internal stability of $F_l(\tilde{R}, P_2)$.

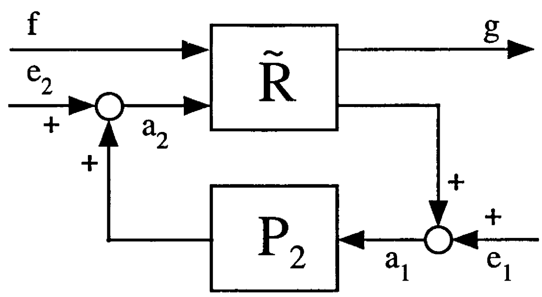

Fig. 5.10: Mapping of $(f' \; e_1' \; e_2')'$ to $(g' \; a_1' \; a_2')'$.

($\Rightarrow$) Let P_1 and P_2 have stabilizable and detectable realizations $(A_{P_1}, B_{P_1}, C_{P_1})$ and $(A_{P_2}, B_{P_2}, C_{P_2}, D_{P_2})$, respectively.[4] From the realization for P_1, construct a realization for $\tilde{R}$ as in (5.6). Using Definition 11(a) from Appendix A, compute the induced realization for $F_l(\tilde{R}, P_2)$ to be

$$\left(\begin{array}{c|c} A_{F_l} & B_{F_l} \\ \hline C_{F_l} & D_{F_l} \end{array} \right),$$

where

$$\begin{aligned} A_{F_l} &:= \begin{pmatrix} A_{\tilde{R}} + B_{\tilde{R}_2} D_{P_2} C_{\tilde{R}_2} & B_{\tilde{R}_2} C_{P_2} \\ B_{P_2} C_{\tilde{R}_2} & A_{P_2} \end{pmatrix} \\ B_{F_l} &:= \begin{pmatrix} B_{\tilde{R}_1} + \sqrt{\gamma^2 - 1} B_{\tilde{R}_2} D_{P_2} \\ \sqrt{\gamma^2 - 1} B_{P_2} \end{pmatrix} \\ C_{F_l} &:= \begin{pmatrix} C_{\tilde{R}_1} + D_{P_2} C_{\tilde{R}_2} & C_{P_2} \end{pmatrix} \end{aligned}$$

[4] It is satisfactory to consider a strictly proper P_1 because Lemma 4 is to be applied to subsequent results where a generalized system F with a realization given as in (5.2), and a nominal system P_0 with an inherited realization (A, B_2, C_2), are considered.

$$D_{F_l} := \sqrt{\gamma^2 - 1} D_{P_2}.$$

This induced realization is stabilizable and detectable since there exist matrices $F_{F_l} = \left(\frac{\gamma}{\sqrt{\gamma^2-1}} B_{P_1}^* X \ \frac{1}{\sqrt{\gamma^2-1}} F_{P_2} \right)$ (where F_{P_2} is a matrix such that $A_{P_2} + B_{P_2} F_{P_2}$ is Hurwitz) and $L_{F_l} = \begin{pmatrix} -\gamma \bar{Y} C_{P_1}^* \\ L_{P_2} \end{pmatrix}$ (where L_{P_2} is a matrix such that $A_{P_2} + L_{P_2} C_{P_2}$ is Hurwitz) such that $A_{F_l} + B_{F_l} F_{F_l}$ and $A_{F_l} + L_{F_l} C_{F_l}$, respectively, are Hurwitz. (Direct substitution shows that $A_{F_l} + L_{F_l} C_{F_l}$ is Hurwitz. Proving that $A_{F_l} + B_{F_l} F_{F_l}$ is Hurwitz is more complicated and the reader is directed to Appendix B.) By Theorem 11 in Appendix A, $\bar{n}(F_l(\tilde{R}, P_2)) = \bar{n}(V)$, where $V(s)$ denotes the transfer function matrix mapping $(f' \ e_1' \ e_2')'$ to $(g' \ a_1' \ a_2')'$ as shown in Fig. 5.10. If $F_l(\tilde{R}, P_2) \in \mathcal{RH}_\infty$, then $\bar{n}(F_l(\tilde{R}, P_2)) = 0$ and so $V \in \mathcal{RH}_\infty$. □

The next result is a consequence of [13, Proposition 1.1]. It guarantees stability of a certain transfer function matrix formed from the LTI system $\tilde{R}$ and a generalized system F.

Corollary 1. *Consider a generalized system $F \in \mathcal{R}^{(p+n)\times(q+m)}$ partitioned as in* (5.1), *with a stabilizable and detectable realization as given by* (5.2). *Suppose that $P_0 := F_u(F, 0) = F_{22}$ has an inherited realization (A, B_2, C_2) from the realization for F, and suppose further that this realization is stabilizable and detectable (see Footnote 2 in Section 5.3). Then*

$$\tilde{R} \star \begin{pmatrix} P_0 & F_{21} \\ F_{12} & F_{11} \end{pmatrix} \tag{5.8}$$

belongs to $\mathcal{RH}_\infty$, where $\tilde{R}$ is defined as in (5.6).

Proof. Note that $\delta_\nu(P_0, P_0) = 0 < \beta$, so by [13, Proposition 1.1], $F_l(\tilde{R}, P_0) \in \mathcal{RH}_\infty$. Next, compute the induced realization for $F_l(\tilde{R}, P_0)$ as in Definition 11(a) of Appendix A, and note that this induced realization is stabilizable and detectable (as was shown for the general case of the induced realization $F_l(\tilde{R}, P_2)$ in the proof of Lemma 4). Since the induced realization for $F_l(\tilde{R}, P_0)$ is stabilizable and detectable, and since $F_l(\tilde{R}, P_0) \in \mathcal{RH}_\infty$, then the '$A$'-matrix of the induced realization for $F_l(\tilde{R}, P_0)$ is Hurwitz. This 'A'-matrix is provided as follows:

$$\begin{pmatrix} A - B_2 B_2^* X - \gamma^2 \bar{Y} C_2^* C_2 & \gamma \bar{Y} C_2^* C_2 \\ -\gamma B_2 B_2^* X & A \end{pmatrix},$$

where X and Z are the stabilizing solutions to the corresponding generalized algebraic Riccati equations and $\gamma := \frac{1}{\beta}$, $Y := \frac{1}{\gamma^2 - 1} Z$ and $\bar{Y} := Y(I - XY)^{-1}$. But computation of the induced realization for (5.8), which is given by

$$\left(\begin{array}{c|cc} A_\star & B_{\star 1} & B_{\star 2} \\ \hline C_{\star 1} & 0 & D_{21} \\ C_{\star 2} & \sqrt{\gamma^2 - 1} D_{12} & D_{11} \end{array} \right), \tag{5.9}$$

where

$$
\begin{aligned}
A_\star &:= \begin{pmatrix} A - B_2B_2^*X - \gamma^2\bar{Y}C_2^*C_2 & \gamma\bar{Y}C_2^*C_2 \\ -\gamma B_2B_2^*X & A \end{pmatrix} \\
B_{\star_1} &:= \begin{pmatrix} \frac{\gamma}{\sqrt{\gamma^2-1}}(I - XY)^{-1}B_2 \\ \sqrt{\gamma^2-1}B_2 \end{pmatrix} \\
B_{\star_2} &:= \begin{pmatrix} \gamma\bar{Y}C_2^*D_{21} \\ B_1 \end{pmatrix} \\
C_{\star_1} &:= \begin{pmatrix} -\gamma C_2 & C_2 \end{pmatrix} \\
C_{\star_2} &:= \begin{pmatrix} -\gamma D_{12}B_2^*X & C_1 \end{pmatrix}
\end{aligned}
$$

(see [104, Chapter 10.4] for state-space formula), shows that $A_\star$ is equal to the 'A'-matrix of the induced realization for $F_l(\tilde{R}, P_0)$. So (5.8) is in $\mathcal{RH}_\infty$. □

Remark 3. *The induced realization for* (5.8)*, as given by* (5.9)*, is stabilizable and detectable since $A_\star$ is Hurwitz.*

We are now in the position to show that the LTI quantity on the LHS of (5.3) can be computed via solving a LMI feasibility problem. Lemma 4 and Corollary 1 are used to obtain an "upper bound" on the LTI quantity on the LHS of (5.3) as stated in Theorem 7 below; then Theorem 7 forms part of the solution algorithm for determining the scaled LTI ν-gap quantity exactly (to within a predetermined tolerance). (This solution algorithm is provided in the next section.) In words, Theorem 7 states the following: if a system of LMI constraints, dependent on the LTI system $\tilde{R}$ and some given number β, is feasible, then the LTI quantity on the LHS of (5.3) is less than or equal to β.

Theorem 7. *Suppose $F \in \mathcal{R}^{(p+n)\times(q+m)}$ is a generalized system partitioned as in* (5.1)*, with a stabilizable and detectable realization as given by* (5.2)*; and suppose $P_0 := F_u(F, 0)$ has an inherited realization (A, B_2, C_2) that is also stabilizable and detectable (see Footnote 2 in Section 5.3). Consider the LTI uncertainty set $\boldsymbol{\delta_o}$ and the set of constant diagonal matrix pairs $\boldsymbol{D}$ defined earlier, and suppose that each $\delta \in \boldsymbol{\delta_o}$ has a given stabilizable and detectable realization and that each induced realization for $F_u(F, D_l^{-1}\delta D_r)$ is stabilizable and detectable (as defined in Appendix A) . Given a $\beta \in (0, b_{opt}(P_0))$, then*

$$
\inf_{\tilde{D}=(D_l,D_r)\in\boldsymbol{D}} \sup_{\delta\in\boldsymbol{\delta_o}} \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r)) \le \beta \tag{5.10}
$$

if $\exists \tilde{D} \in \boldsymbol{D} : \forall\omega \in \mathbb{R}\ \exists d_\omega \in \mathbb{R}_+ :$

$$
J^*(j\omega)\begin{pmatrix} d_\omega^2 I_n & 0 \\ 0 & D_r^2 \end{pmatrix} J(j\omega) < \begin{pmatrix} d_\omega^2 I_m & 0 \\ 0 & D_l^2 \end{pmatrix},
$$

where $J := \tilde{R} \star \left(\begin{smallmatrix} P_0 & F_{21} \\ F_{12} & F_{11} \end{smallmatrix}\right)$, *and* $\tilde{R} \in \mathcal{R}^{(n+m)\times(m+n)}$ *is defined as in* (5.6).

Proof. Define $H := \left(\begin{smallmatrix} 0 & I_n \\ I_p & 0 \end{smallmatrix}\right) F \left(\begin{smallmatrix} 0 & I_q \\ I_m & 0 \end{smallmatrix}\right)$ so that $J = \tilde{R} \star H$ and $F_u(F, D_l^{-1}\delta D_r) = F_l(H, D_l^{-1}\delta D_r)$. Then

$$\begin{aligned}
&\inf_{\tilde{D}=(D_l,D_r)\in \boldsymbol{D}} \sup_{\delta\in\boldsymbol{\delta_o}} \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r)) \leq \beta \\
&\Leftarrow \exists \tilde{D} \in \boldsymbol{D} : \forall \delta \in \boldsymbol{\delta_o}\ \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r)) \leq \beta \\
&\Leftrightarrow \exists \tilde{D} \in \boldsymbol{D} : \forall \delta \in \boldsymbol{\delta_o}\ \|F_l(\tilde{R}, F_l(H, D_l^{-1}\delta D_r))\|_\infty \leq 1 \text{ and the system in Fig. 5.11} \\
&\quad \text{is internally stable.}
\end{aligned} \tag{5.11}$$

The last equivalence of the above statement is true due to the following. From Lemma 4 we know that

$$\begin{gathered}
\delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r)) \leq \beta \\
\Updownarrow \\
F_l(\tilde{R}, F_l(H, D_l^{-1}\delta D_r))\|_\infty \leq 1 \text{ and } V \in \mathcal{RH}_\infty,
\end{gathered}$$

where $V(s)$ denotes the transfer function matrix mapping $(f'\ e_1'\ e_2')'$ to $(g'\ a_1'\ a_2')'$ as shown in Fig. 5.12. Now construct a realization for $\tilde{R}$ as in (5.6); let a stabilizable and detectable realization for a $D_l^{-1}\delta D_r$ be given by $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$ and let an induced realization for a $F_l(H, D_l^{-1}\delta D_r)$ (which is stabilizable and detectable due to the supposition in the theorem statement) be given by

$$\left(\begin{array}{c|c} A_{\bar\eta} & B_{\bar\eta} \\ \hline C_{\bar\eta} & D_\eta \end{array}\right),$$

where $A_{\bar\eta} := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) A_\eta \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, $B_{\bar\eta} := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) B_\eta$, $C_{\bar\eta} := C_\eta \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and A_η, B_η, C_η are given in Definition 11(b) of Appendix A. Then $V \in \mathcal{RH}_\infty$ if and only if the 'A'-matrix of the induced realization for $V(s)$, given by

$$\left(\begin{array}{c|c} A_V & B_V \\ \hline C_V & D_V \end{array}\right)$$

where

$$\begin{aligned}
A_V &:= \begin{pmatrix} A_{\tilde R} + B_{\tilde R_2} D_\eta C_{\tilde R_2} & B_{\tilde R_2} C_{\bar\eta} \\ B_{\bar\eta} C_{\tilde R_2} & A_{\bar\eta} \end{pmatrix} \\
B_V &:= \begin{pmatrix} B_{\tilde R_1} + \sqrt{\gamma^2-1} B_{\tilde R_2} D_\eta & B_{\tilde R_2} D_\eta & B_{\tilde R_2} \\ \sqrt{\gamma^2-1} B_{\bar\eta} & B_{\bar\eta} & 0 \end{pmatrix} \\
C_V &:= \begin{pmatrix} C_{\tilde R_1} + D_\eta C_{\tilde R_2} & C_{\bar\eta} \\ C_{\tilde R_2} & 0 \\ D_\eta C_{\tilde R_2} & C_{\bar\eta} \end{pmatrix} \\
D_V &:= \begin{pmatrix} \sqrt{\gamma^2-1} D_\eta & D_\eta & I \\ \sqrt{\gamma^2-1} I & I & 0 \\ \sqrt{\gamma^2-1} D_\eta & D_\eta & I \end{pmatrix},
\end{aligned}$$

is Hurwitz [32, Lemma A.4.1].

Now construct the induced realization for J as in (5.9), noting that stabilizability and detectability of this realization is guaranteed (see Remark 3). Let $\bar{U}(s)$ denote the transfer function matrix mapping $(f'\ d_1'\ d_2')'$ to $(g'\ z'\ w')'$ as shown in Fig. 5.13. It is true that $\bar{U} \in \mathcal{RH}_\infty$ if and only if the 'A'-matrix of the induced realization for $\bar{U}(s)$, given by

$$\left(\begin{array}{c|c} A_{\bar{U}} & B_{\bar{U}} \\ \hline C_{\bar{U}} & D_{\bar{U}} \end{array}\right)$$

where

$$A_{\bar{U}} := \begin{pmatrix} A_\star + B_{\star_2} Q \breve{D} C_{\star_2} & B_{\star_2} Q \breve{C} \\ \breve{B} R C_{\star_2} & \breve{A} + \breve{B} R D_{11} \breve{C} \end{pmatrix}$$

$$B_{\bar{U}} := \begin{pmatrix} B_{\star_1} + \sqrt{\gamma^2 - 1} B_{\star_2} Q \breve{D} D_{12} & B_{\star_2} Q \breve{D} & B_{\star_2} Q \\ \sqrt{\gamma^2 - 1} \breve{B} R D_{12} & \breve{B} R & \breve{B} R D_{11} \end{pmatrix}$$

$$C_{\bar{U}} := \begin{pmatrix} C_{\star_1} + D_{21} Q \breve{D} C_{\star_2} & D_{21} Q \breve{C} \\ R C_{\star_2} & R D_{11} \breve{C} \\ Q \breve{D} C_{\star_2} & Q \breve{C} \end{pmatrix}$$

$$D_{\bar{U}} := \begin{pmatrix} \sqrt{\gamma^2 - 1} D_{21} Q \breve{D} D_{12} & D_{21} Q \breve{D} & D_{21} Q \\ \sqrt{\gamma^2 - 1} R D_{12} & R & R D_{11} \\ \sqrt{\gamma^2 - 1} Q \breve{D} D_{12} & Q \breve{D} & Q \end{pmatrix},$$

$R := (I - D_{11}\breve{D})^{-1}$ and $Q := (I - \breve{D} D_{11})^{-1}$, is Hurwitz. A simple calculation shows that $A_V = A_{\bar{U}}$.

Note that

$$F_l(\tilde{R}, F_l(H, D_l^{-1}\delta D_r)) = F_l(J, D_l^{-1}\delta D_r) = F_u(G, \delta),$$

where $G := \left(\begin{smallmatrix} 0 & D_r \\ I_n & 0 \end{smallmatrix}\right) J \left(\begin{smallmatrix} 0 & I_m \\ D_l^{-1} & 0 \end{smallmatrix}\right)$. This means that (5.11) holds

$\Leftrightarrow \exists \widetilde{D} \in \boldsymbol{D} : \forall \delta \in \boldsymbol{\delta_o}\ ||F_u(G, \delta)||_\infty \leq 1$ and the system in Fig. 5.11 is internally stable

$$\Leftrightarrow \exists \widetilde{D} \in \boldsymbol{D} : \sup_{\omega \in \mathbb{R}} \mu_{\boldsymbol{\delta_s}}(G(j\omega)) \leq 1, \tag{5.12}$$

where μ is the structured singular value with respect to the structured set $\boldsymbol{\delta_s} := \left\{ \begin{pmatrix} \delta & 0 \\ 0 & \hat{\delta} \end{pmatrix} : \hat{\delta} \in \mathcal{RH}_\infty^{m \times n}, \|\hat{\delta}\|_\infty < 1, \delta \in \boldsymbol{\delta_o} \right\}$, using [104, Theorem 11.9] and noting that $J \in \mathcal{RH}_\infty$ from Corollary 1. Finally, let $d_\omega \in \mathbb{R}_+$ such that

$$\begin{pmatrix} I_q & 0 \\ 0 & d_\omega I_m \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & \hat{\delta} \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & \frac{1}{d_\omega} I_n \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & \hat{\delta} \end{pmatrix}$$

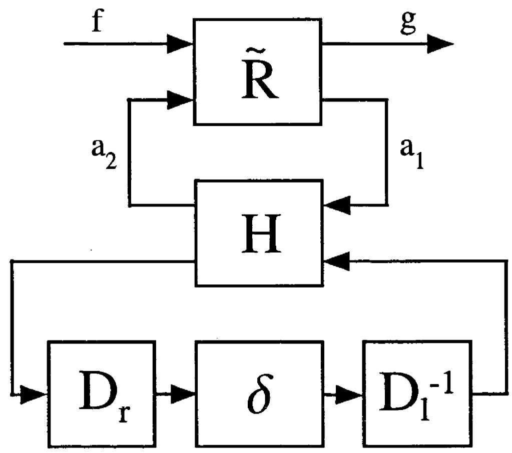

Fig. 5.11: Internal stability of $F_l(J, D_l^{-1}\delta D_r)$.

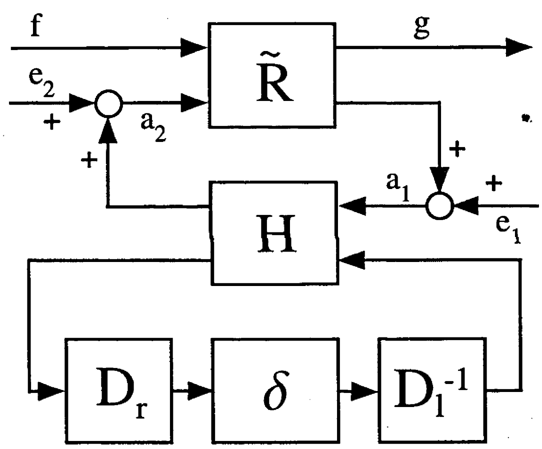

Fig. 5.12: Mapping of $(f'\ e_1'\ e_2')'$ to $(g'\ a_1'\ a_2')'$.

for all $\begin{pmatrix} \delta & 0 \\ 0 & \hat{\delta} \end{pmatrix} \in \boldsymbol{\delta_s}$. At each frequency ω, $\mu_{\boldsymbol{\delta_s}}(G(j\omega))$ is equal to

$$\inf_{d_\omega \in \mathbb{R}_+} \bar{\sigma}\left(\begin{pmatrix} 0 & D_r \\ d_\omega I_n & 0 \end{pmatrix} J(j\omega) \begin{pmatrix} 0 & \frac{1}{d_\omega} I_m \\ D_l^{-1} & 0 \end{pmatrix} \right)$$

from [104, Theorem 11.5], where $\bar{\sigma}$ denotes the maximum singular value. So (5.12)

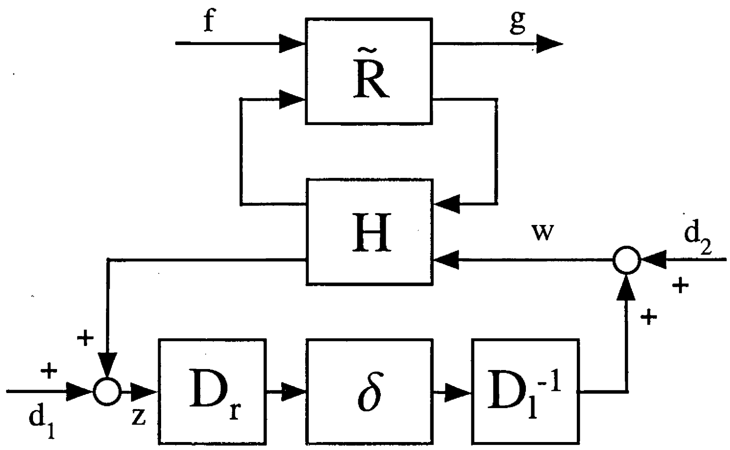

Fig. 5.13: Mapping of $(f' \; d_1' \; d_2')'$ to $(g' \; z' \; w')'$.

holds

$$\Leftarrow \exists \widetilde{D} \in \boldsymbol{D} : \forall \omega \in \mathbb{R} \; \exists d_\omega \in \mathbb{R}_+ : \bar{\sigma}\left(\begin{pmatrix} 0 & D_r \\ d_\omega I_n & 0 \end{pmatrix} J(j\omega) \begin{pmatrix} 0 & \frac{1}{d_\omega} I_m \\ D_l^{-1} & 0 \end{pmatrix} \right) \leq 1$$

$$\Leftrightarrow \exists \widetilde{D} \in \boldsymbol{D} : \forall \omega \in \mathbb{R} \; \exists d_\omega \in \mathbb{R}_+ : J^*(j\omega) \begin{pmatrix} d_\omega^2 I_n & 0 \\ 0 & D_r^2 \end{pmatrix} J(j\omega) \leq \begin{pmatrix} d_\omega^2 I_m & 0 \\ 0 & D_l^2 \end{pmatrix}.$$

□

The implications that are not necessary and sufficient in the above proof could be easily tightened to necessary and sufficient implications by replacing several $\leq$ with $<$ and considering the closed set $\boldsymbol{\delta}$ and $\mathbb{R} \cup \{\infty\}$ where appropriate, similar to the proof of Theorem 6. However, one key equivalence in the above proof relating a ν-gap ball to a $\mathcal{H}_\infty$-ball (see Lemma 4 based on [13, Proposition 1.1]) is only stated in terms of non-strict inequalities. It appears that it may be possible to rewrite Lemma 4 with strict inequalities, thereby allowing for a tightening of Theorem 7 so that it is necessary and sufficient; but this has not been investigated in this dissertation. At this stage, it is simply pointed out that the condition in Theorem 7, as written, is "close to" necessary for (5.10) to hold, as necessity is only lost at closure of sets $\boldsymbol{\delta_o}, \mathbb{R}, \boldsymbol{D}$.

5.5 Solution Algorithm

A solution algorithm that can be used to determine the exact scaled LTI quantity on the LHS of (5.3) is now provided. The solution algorithm itself is based on a standard bisectional search. The notion is to use Theorem 7 to determine feasibility of a system of LMI constraints with respect to a test value β. Iterations of the bisectional line search are implemented over the interval $(0, b_{opt}(P_0))$ to select subsequent test values for β. The direction in which the line search proceeds depends on the 'true' or 'false' result acquired by solving the LMI feasibility problem: a 'false' result suggests that a larger test β should be chosen; while a 'true' result indicates one can try a smaller test β. Consequently, the LTI quantity $\inf_{\tilde{D}=(D_l,D_r)\in \boldsymbol{D}} \sup_{\delta\in\delta_o} \delta_\nu(P_0, F_u(F, D_l^{-1}\delta D_r))$ is achieved to within a sufficiently small pre-determined tolerance. Provided that the LTI quantity obtained is smaller than the generalized robust stability margin $b_{P_0,K}$ achieved with some controller K that internally stabilizes the nominal plant, then internal stability of the system $[P_{LTV}, K]$ for all time-varying perturbations $\Delta \in \boldsymbol{\Delta}$ is guaranteed.

The complete solution algorithm is as follows:

1) Set the bounds on possible β to be $\alpha_l = 0$ and $\alpha_r = b_{opt}(P_0)$. Set a sufficiently small tolerance $\epsilon > 0$ for the iterative bisections with respect to finding β to end. Select an initial $\beta_0 = \alpha_r - \epsilon$ and set $\beta_{feas} = b_{opt}(P_0)$. Set $i = 0$. Goto Step 2.

2) Given a β_i, solve the convex optimization problem: "does there exist a $\tilde{D} \in \boldsymbol{D}$ such that, for each $\omega \in \mathbb{R}$, there exists a corresponding $d_\omega \in \mathbb{R}_+$ such that

$$J^*(j\omega)\begin{pmatrix} d_\omega^2 I_n & 0 \\ 0 & D_r^2 \end{pmatrix} J(j\omega) \leq \begin{pmatrix} d_\omega^2 I_m & 0 \\ 0 & D_l^2 \end{pmatrix},$$

where $J := \tilde{R} \star \begin{pmatrix} P_0 & F_{21} \\ F_{12} & F_{11} \end{pmatrix}$, and $\tilde{R}$ is defined as in (5.6)". Now,

   i) If the optimization problem is feasible, set $\beta_{feas} = \beta_i$ and $\beta_{i+1} = \frac{\alpha_l+\beta_i}{2}$. Update $\alpha_r = \beta_i$. Goto Step 2iii.

   ii) If the optimization problem is not feasible, test if $\beta_{feas} - \beta_i \leq \epsilon$. If yes, then end. If no, set $\beta_{i+1} = \frac{\beta_i+\alpha_r}{2}$. Update $\alpha_l = \beta_i$. Goto Step 2iii.

   iii) Set $i = i + 1$ and goto Step 2.

If $\beta_{feas} < b_{P_0,K}$, where K is some internally stabilizing controller, then $[P_{LTV}, K]$ is internally stable for all $\Delta \in \boldsymbol{\Delta}$. If not, internal stability of $[P_{LTV}, K]$ has not been determined (and a possibility if $\beta_{feas} \neq b_{opt}(P_0)$ is to choose a different controller to obtain a larger stability margin).

The convex optimization problem in Step 2 of the solution algorithm is easily solved using Matlab's LMI toolbox for instance. A numerical example follows in the next section for completeness.

5.6 Numerical Example

The following example illustrates the implementation of the solution algorithm. The example has been taken from [61]. In [61], an optimization problem that integrates a number of steps of the standard $\mathcal{H}_\infty$ loop-shaping design procedure [65] is introduced. The idea is to maximize the generalized robust stability margin of the shaped plant $P_s = W_2 P W_1$, where P is the scaled nominal open-loop plant, over allowable loop-shaping weights W_1 and W_2, while ensuring that the resulting loop-shape lies in a pre-defined region that characterizes the desired performance specifications (see Fig. 5.14).

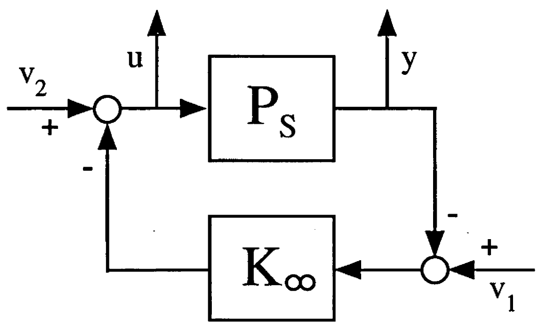

Fig. 5.14: $\mathcal{H}_\infty$ loop-shaping framework.

The plant used in the example in [61] is a scaled-down version of the high incidence research model developed by the Defence Evaluation and Research Agency in Bedford, UK. A physical model of this was constructed at the University of Cambridge in order to investigate problems associated with the control of air-vehicles at high angles of attack. Details of the experiment carried out on this plant may be found in [72].

Input data for the following was given in [61, Section 5]: the scaled nominal open-loop plant P; the loop-shape boundaries; and the loop-shape weight singular value and condition number bounds. Implementation of the algorithm presented in [61] (for the case in which a diagonal pre-compensator W_1 is required and the post-compensator W_2 is held fixed) produced the maximized value of $b_{opt}(P_s)$, the loop-shaping weights W_1 and W_2 that achieved this maximized robust stability margin, and a robustly (in terms of stable LTI perturbations to the coprime factors of P_s) stabilizing controller K_∞ as output. In particular, the shaped plant $P_s = W_2 P W_1$ was found to be given by the state-space model shown in Fig. 5.15, and $b_{opt}(P_s)$ was found to be 0.376 using Matlab's μ-Analysis and Synthesis Toolbox "ncfsyn" function.

$$
\left(\begin{array}{cccccccccccccccc|cc}
0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-23.8 & -3.36 & 4.60 & -0.239 & 11.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-16.8 & -0.0248 & 22.8 & -0.916 & 0 & 0 & 9.83 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -50.0 & 63.2 & 0 & 0 & 112 & -9590 & -1530 & -1250 & 0 & 0 & 0 & 0 & -45300 & 0 \\
0 & 0 & 0 & 0 & 0 & -40.0 & 0 & 0 & -142 & 12100 & 1930 & 1580 & 0 & 0 & 0 & 0 & 57400 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -50.0 & 63.2 & 0 & 0 & 0 & 0 & -58.0 & 10600 & -145 & -1320 & 0 & -55500 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -40.0 & 0 & 0 & 0 & 0 & 73.3 & -13400 & 183 & 1670 & 0 & 70200 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.00561 & 1.14 & 0.178 & 0.123 & 0 & 0 & 0 & 0 & -15.9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.14 & -258 & -43.9 & -74.7 & 0 & 0 & 0 & 0 & -1360 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.178 & -43.9 & -7.52 & -13.9 & 0 & 0 & 0 & 0 & -216 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.123 & 74.7 & 13.9 & -43.9 & 0 & 0 & 0 & 0 & 177 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.00271 & 1.52 & -0.0158 & -0.120 & 0 & 8.20 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.52 & -262 & 5.05 & 72.0 & 0 & 1500 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0158 & 5.05 & -0.117 & -2.76 & 0 & -20.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.120 & -72.0 & 2.76 & -43.9 & 0 & 187 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Fig. 5.15: State-space model of P_s.

Suppose a control system designer wanted to determine to what extent the feedback interconnection shown in Fig. 5.14 would remain internally stable in the face of LTV uncertainty. For instance, consider the uncertain shaped plant shown in Fig. 5.16, where Δ_1 and Δ_2 represent output multiplicative and input feedback LTV uncertainties, respectively. Formally, the uncertain shaped plant model shown in Fig. 5.16 is described by

$$(I + \epsilon_1 \Delta_1) P_s (I - \epsilon_2 \Delta_2)^{-1}, \tag{5.13}$$

where $\Delta_1, \Delta_2 \in \mathbf{\Delta}$ and $\epsilon_1, \epsilon_2 \in [0, 1]$. Recall that output multiplicative uncertainty may typically represent output (sensor) errors or neglected high frequency dynamics, while input feedback uncertainty may represent low frequency parameter errors (see [104, Table 9.1]). Expressing (5.13) in the standard structured uncertainty form gives

$$F = \begin{pmatrix} 0 & \epsilon_1\epsilon_2 P_s & \epsilon_1 P_s \\ 0 & \epsilon_2 I & I \\ I & \epsilon_2 P_s & P_s \end{pmatrix}$$

(and so F is a transfer function matrix which relates the structure of the uncertainties Δ_1 and Δ_2 to P_s).

The solution algorithm presented in this chapter can be, and was, used to determine stability robustness of the uncertain feedback interconnection. The input to the algorithm consisted of the state-space model of the shaped plant P_s as given in Fig. 5.15, a (stabilizable and detectable) state-space model for the transfer function matrix F such that the state-space model for P_s was inherited from F and the realizations for F_{21} and F_{12} had no unstable invariant zeros, and the size of the scaling

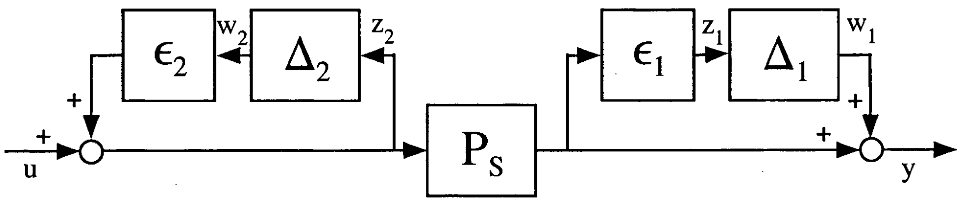

Fig. 5.16: Open-loop shaped plant with uncertainty.

factors ϵ_1 and ϵ_2. The algorithm was coded up in Matlab 6.5. One hundred equally spaced frequency points on a logarithmic scale between $\omega = 10^{-4}$ and 10^4 rad/s were chosen for Step 2 of the algorithm and a tolerance of 0.001 was chosen for Step 1.

First, stability robustness of the system subject only to output multiplicative uncertainty was investigated. Four hundred and one evenly spaced scaling factors ϵ_1 were chosen from between $[0, 1]$ to represent different sizes of the uncertainty, while ϵ_2 was set fixed at zero. An algorithm output quantity β_{feas} (representative of the LTI quantity on the LHS of (5.3)) was produced for each of the 401 pairs of uncertainty scaling factors (ϵ_1, ϵ_2). The results for where $\epsilon_1 \in [0, 0.5]$ are shown in Fig. 5.17. For example, a size of $\epsilon_1 = 0.4975$ resulted in a β_{feas} of 0.367, which is less than $b_{opt}(P_s) = 0.376$. This means that the interconnection $[P_s, K_\infty]$ subject to LTV output multiplicative uncertainties with scaling factors of size up to and including 0.4975 as described by (5.13), will be internally stable. Note that the next (larger) scaling factor tested was $\epsilon_1 = 0.5$, for which the algorithm produced an output $\beta_{feas} > b_{opt}(P_s)$ and so internal stability of $[P_s, K_\infty]$ subject to LTV output multiplicative uncertainties with $\epsilon_1 > 0.4975$ was not concluded here.

Next, stability robustness of the system subject only to input feedback uncertainty was investigated. The scaling factor ϵ_1 was set fixed at zero and β_{feas} was computed with respect to 401 evenly spaced input feedback uncertainty scaling factors ϵ_2 ranging from between $[0, 1]$. The results for where ϵ_2 ranged between $[0, 0.55]$ are shown in Fig. 5.18. Here, a size of $\epsilon_2 = 0.5275$ resulted in a LTI quantity of 0.374, which is less than $b_{opt}(P_s)$, and so $[P_s, K_\infty]$ subject to LTV input feedback uncertainties of size less than or equal to 0.5275 as described by (5.13) was guaranteed to be internally stable. Again, internal stability when $\epsilon_2 > 0.5275$ could not be concluded.

Finally, stability robustness of the feedback interconnection was tested with respect to when P_s was subjected to both output multiplicative and input feedback LTV uncertainties. For example, when the scaling factors were set to $\epsilon_1 = 0.35$ and

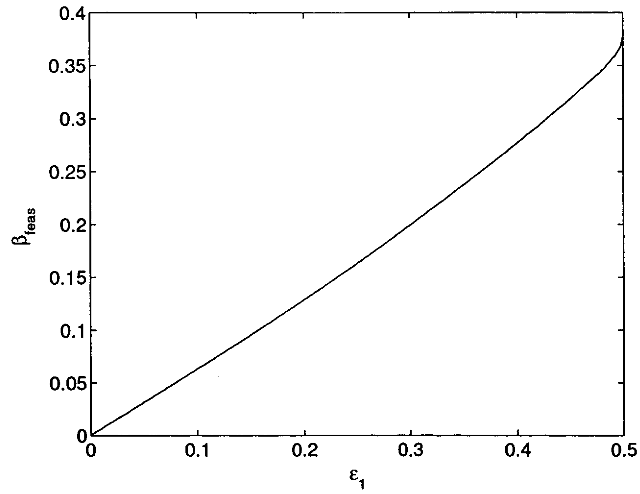

Fig. 5.17: The quantity β_{feas} with respect to the size of the output multiplicative uncertainty.

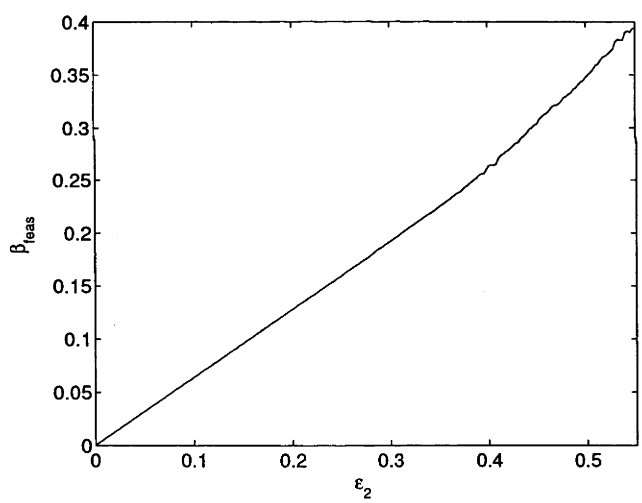

Fig. 5.18: The quantity β_{feas} with respect to the size of the input feedback uncertainty.

$\epsilon_2 = 0.38$, the algorithm produced a β_{feas} of 0.371, meaning that $[P_s, K_\infty]$ subject to both output multiplicative and input feedback LTV uncertainties of size 0.35 and 0.38, respectively, as described by (5.13), is guaranteed to be internally stable.

5.7 Conclusions

The scaled small gain result, traditionally used to determine stability robustness of LTI nominal systems subject to structured LTV uncertainty, was extended into a scaled LTI ν-gap metric framework. It was shown that the scaled LTI ν-gap metric condition can be checked by solving a LMI feasibility problem (as can the scaled small gain condition). Furthermore, checking the scaled LTI ν-gap metric condition requires dependence on the controller only in terms of computing a generalized robust stability margin (as opposed to the scaled small gain condition, where dependence on K occurs in the LMI feasibility problem set-up).

6. A SPECIAL CASE OF THE SCALED LTI ν-GAP METRIC CONDITION[1]

6.1 Introduction

In the previous chapter, a scaled LTI ν-gap metric condition was introduced as a tool for determining the stability robustness of a closed-loop system subject to LTV uncertainty. It was shown that the scaled LTI ν-gap metric condition can be checked by implementing a solution algorithm based on a LMI feasibility problem. In this chapter, a case in which the (LHS of the) scaled LTI ν-gap metric condition is analytically computable is investigated.

In particular, the case under investigation is one where all plants considered are SISO, and the LTV uncertainties are output-multiplicative in nature (as shown in Fig. 6.1). Examples of such classes of systems may arise in practice when one has input saturation nonlinearities or output sensor nonlinearities.

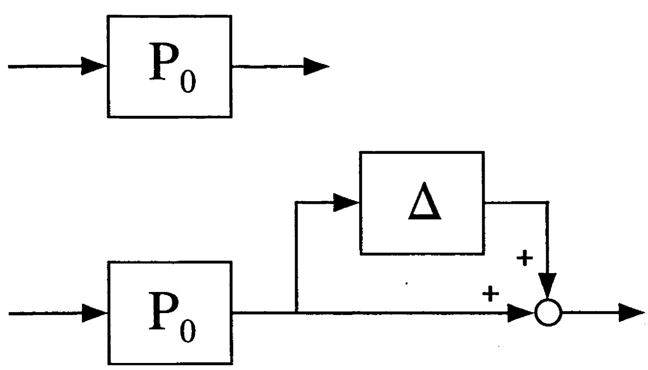

Fig. 6.1: Nominal open-loop LTI system, and nominal open-loop LTI system subject to LTV output-multiplicative uncertainty.

[1] Content from this chapter was published in [38].

6.2 Problem Set-up

We begin by setting the stability robustness problem up as it was done in Chapter 5. Suppose that P_0 is a strictly proper LTI plant. Let $\boldsymbol{\Delta}$ denote the set of all causal (not necessarily memoryless), stable, block-diagonal, LTV uncertainties with $\mathcal{L}_2$-induced norm less than or equal to one. (For now, suppose that each block is a square LTV operator of dimension $p_i \times p_i$.) We will use the notation Δ_ϵ when we are particularly interested in considering a LTV uncertainty with $\mathcal{L}_2$-induced norm less than or equal to ϵ (that is, uncertainty that has not been normalized). Let K denote a nominally stabilizing controller.

The closed-loop stability robustness problem is shown in Fig. 6.2; and, of course, can be recast into the framework shown in Fig. 6.3, where $F = \left(\begin{smallmatrix} 0 & \epsilon P_0 \\ I & P_0 \end{smallmatrix}\right)$ is a transfer function matrix relating the uncertainty and the nominal plant. Letting $Z := \mathcal{F}_l(F, K) \in \mathcal{RH}_\infty$, the system shown in Fig. 6.3 is further reduced to the system shown in Fig. 6.4. As in Chapter 5, associate with $\boldsymbol{\Delta}$ a set of scalings that commute with the set of perturbations. In particular, choose $\boldsymbol{D} = \{\tilde{D} : \tilde{D} = \operatorname{diag}(d_1 I_{p_1}, d_2 I_{p_2}, \ldots, d_n I_{p_n}), d_i \in \mathbb{R}, d_i > 0\}$ such that $\tilde{D}^{-1}\Delta\tilde{D} = \Delta$ for all $\Delta \in \boldsymbol{\Delta}$. Consider the closed-loop system shown in Fig. 6.5. It follows that the stability robustness condition for Z and $\tilde{D}^{-1}Z\tilde{D}$ is the same for any $\tilde{D} \in \boldsymbol{D}$.

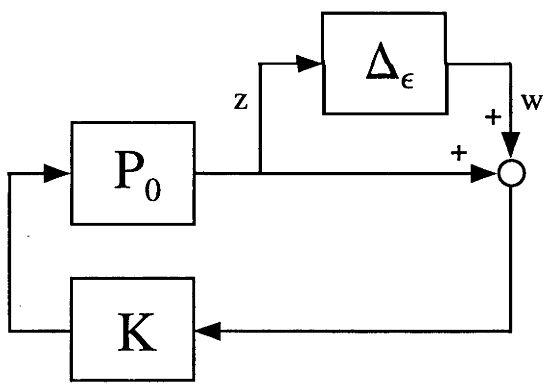

Fig. 6.2: Closed-loop stability robustness problem.

At this point, we recall the scaled small gain condition for robust stability that was described in Chapter 5.

Theorem 8. *Given a causal, LTI system with transfer function matrix $Z \in \mathcal{RH}_\infty$, the system shown in Fig. 6.4 is stable for all $\Delta \in \boldsymbol{\Delta}$ if and only if* $\inf_{\tilde{D} \in \boldsymbol{D}} \|\tilde{D}^{-1}Z\tilde{D}\|_\infty < 1$.

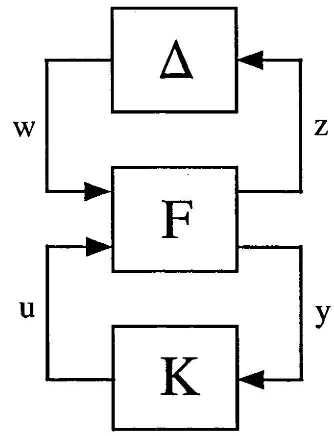

Fig. 6.3: General framework.

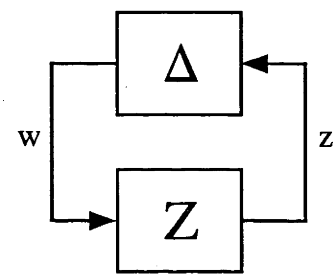

Fig. 6.4: Reduced stability robustness problem.

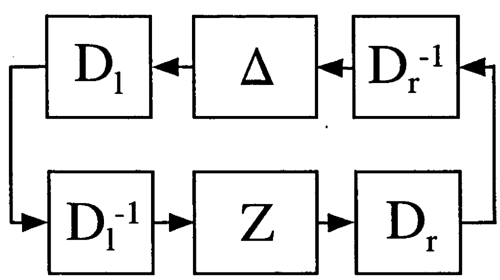

Fig. 6.5: Re-scaled closed-loop system.

See [20,80] for the proof. Note that if the set of scalings is not restricted to those that commute with the set of perturbations, but rather only with the norm of the perturbations (ie: $\|\tilde{D}^{-1}\Delta\tilde{D}\| = \|\Delta\|$), then the necessary and sufficient condition also holds where Δ is nonlinear and time-invariant [17]. We do not consider this case further though. Since we intend to consider SISO systems in this chapter, the D-scales are now dropped from the formulation and hence the necessary and sufficient condition for robust stability in Theorem 8 becomes simply $\|Z\|_\infty < 1$ (which is the standard small gain condition).

Now, suppose that a LTI closed-loop system as shown in Fig. 6.6, is similarly recast as a robust stability problem as shown in Fig. 6.4 (where δ such that $\delta \in \mathcal{RH}_\infty$ and $\|\delta\|_\infty \leq 1$ replaces Δ). Then the closed-loop LTI system has the same small gain condition for robust stability associated with it as the LTV closed-loop system (since the closed-loop LTI system is internally stable for all $\delta \in \mathcal{RH}_\infty$ such that $\|\delta\|_\infty \leq 1$ if and only if $\|Z\|_\infty < 1$) [104]. We thus have Lemma 5.

Lemma 5. *Stability, for all $\delta \in \mathcal{RH}_\infty$ such that $\|\delta\|_\infty \leq 1$, of the LTI system shown in Fig. 6.6 is equivalent to stability, for all Δ_ϵ, of the LTV system shown in Fig. 6.2 (where Δ_ϵ is causal, stable, block-diagonal, LTV uncertainty with $\mathcal{L}_2$-induced norm less than or equal to ϵ).*

Proof. Since both δ and Δ are SISO perturbations, $\|Z\|_\infty < 1$ is a necessary and sufficient condition for both the first and second parts of the lemma statement. □

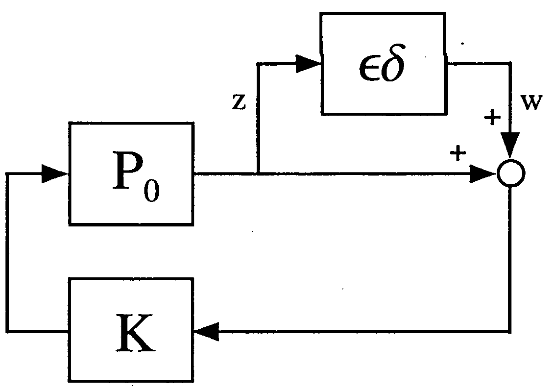

Fig. 6.6: Closed-loop LTI stability robustness problem.

6.3 A "Worst-case" LTI ν-gap Metric

The stability problem concerning the LTI system shown in Fig. 6.6 is now recast in terms of the LTI ν-gap metric. We assume that realizations for Δ_ϵ, δ and K, and the induced realizations for $(1+\Delta_\epsilon)P_0$, $(1+\epsilon\delta)P_0$ and Z, are stabilizable and detectable. Stabilizability and detectability of induced realizations are discussed in Appendix A. Note that concepts of "stabilizability" and "detectability", as associated with the LTV uncertainty Δ_ϵ, are traditionally referred to as uniform stabilizability and uniform detectability respectively (see Section A.3).

From [90] (and Chapter 2), it is known that if the distance between P_0 and the perturbed plant $(1+\epsilon\delta)P_0$, as measured in the ν-gap metric, is sufficiently small in the generalized robust stability margin sense for all $\delta \in \mathcal{RH}_\infty$ such that $\|\delta\|_\infty \leq 1$, then the interconnection $[(1+\epsilon\delta)P_0, K]$ is internally stable for all $\delta \in \mathcal{RH}_\infty$ such that $\|\delta\|_\infty \leq 1$. That is, if

$$\delta_\nu(P_0, (1+\epsilon\delta)P_0) < b_{P_0,K}$$

$\forall\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1$, then $[(1+\epsilon\delta)P_0, K]$ is stable $\forall\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1$. Clearly, this statement can be rewritten as follows: if

$$\sup_{\delta\in\mathcal{RH}_\infty:\|\delta\|_\infty\leq 1} \delta_\nu(P_0, (1+\epsilon\delta)P_0) < b_{P_0,K}, \tag{6.1}$$

then $[(1+\epsilon\delta)P_0, K]$ is stable for all $\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1$.

Using the SISO chordal distance formula for the LTI ν-gap metric provided in [86], which has the form

$$\kappa(P_1(j\omega), P_2(j\omega)) = \frac{|P_2(j\omega) - P_1(j\omega)|}{\sqrt{1+|P_1(j\omega)|^2}\sqrt{1+|P_2(j\omega)|^2}}, \tag{6.2}$$

the quantity $\sup_{\delta\in\mathcal{RH}_\infty:\|\delta\|_\infty\leq 1} \delta_\nu(P_0, (1+\epsilon\delta)P_0)$ can be reformulated into an analytically computable form for which the winding number condition is always satisfied, as follows.

Theorem 9. *Given $\epsilon \in [0,1)$,*

$$\sup_{\delta\in\mathcal{RH}_\infty:\|\delta\|_\infty\leq 1} \delta_\nu(P_0, (1+\epsilon\delta)P_0) = \delta_\nu(P_0, (1-\epsilon)P_0). \tag{6.3}$$

Proof. A lower bound is first placed on the LHS of (6.3) as follows by choosing $\delta = -1$:

$$\sup_{\delta\in\mathcal{RH}_\infty:\|\delta\|_\infty\leq 1} \delta_\nu(P_0, (1+\epsilon\delta)P_0) \geq \delta_\nu(P_0, (1-\epsilon)P_0).$$

To place an upper bound on the LHS of (6.3), the winding number condition for the LTI ν-gap metric must first be checked (since, if it is not satisfied for some

$\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1$, then the metric will take the value of 1 at this δ and hence so will the LHS of (6.3)). Let G be a normalized graph symbol for the nominal plant P_0.[2] Then $\left(\begin{smallmatrix} 1+\epsilon\delta & 0 \\ 0 & 1 \end{smallmatrix}\right) GQ^{-1}$ is a normalized graph symbol for the perturbed plant $(1+\epsilon\delta)P_0$ if Q is a unit in $\mathcal{RH}_\infty$ satisfying $Q^\sim Q = G^\sim \left(\begin{smallmatrix} (1+\epsilon\delta^\sim)(1+\epsilon\delta) & 0 \\ 0 & 1 \end{smallmatrix}\right) G$. Then

$$\begin{aligned}
\operatorname{wno}\det(G^* \left(\begin{smallmatrix} 1+\epsilon\delta & 0 \\ 0 & 1 \end{smallmatrix}\right) GQ^{-1}) &= \operatorname{wno}\det((1+\epsilon\delta N^* N)Q^{-1}) \\
&= \operatorname{wno}\det(1+\epsilon\delta N^* N) - \operatorname{wno}\det(Q) \\
&= \operatorname{wno}\det(1+\epsilon\delta N^* N)
\end{aligned}$$

(since Q is a unit in $\mathcal{RH}_\infty$). Furthermore, $\|N^* N\|_\infty \leq \|N\|_\infty^2 \leq \|\left(\begin{smallmatrix} N \\ M \end{smallmatrix}\right)\|_\infty^2 = 1$; and hence $\|\epsilon\delta N^* N\|_\infty \leq \epsilon\|\delta\|_\infty \|N^* N\|_\infty < 1$ for all $\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1$ since $\epsilon \in [0,1)$. So

$$\operatorname{wno}\det(1+\epsilon\delta N^* N) = 0,$$

using [86, Equation (1.9)]. Thus, for all $\delta \in \mathcal{RH}_\infty$ such that $\|\delta\|_\infty \leq 1$, the winding number condition is satisfied; and hence $\delta_\nu(P_0, (1+\epsilon\delta)P_0) = \sup_\omega \kappa(P_0(j\omega), (1+\epsilon\delta(j\omega))P_0(j\omega))$ for all $\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1$.

An upper bound is now placed on the LHS of (6.3) as follows. For all $\delta \in \mathcal{RH}_\infty$ such that $\|\delta\|_\infty \leq 1$, we have

$$\begin{aligned}
\kappa(P_0(j\omega), (1+\epsilon\delta(j\omega))P_0(j\omega)) &= \frac{|P_0(j\omega) - (1+\epsilon\delta)P_0(j\omega)|}{\sqrt{1+|P_0(j\omega)|^2}\sqrt{1+|1+\epsilon\delta|^2|P_0(j\omega)|^2}} \\
&= \frac{\epsilon|\delta||P_0(j\omega)|}{\sqrt{1+|P_0(j\omega)|^2}\sqrt{1+|1+\epsilon\delta|^2|P_0(j\omega)|^2}} \\
&\leq \frac{\epsilon|P_0(j\omega)|}{\sqrt{1+|P_0(j\omega)|^2}\sqrt{1+|1+\epsilon\delta|^2|P_0(j\omega)|^2}} \\
&\leq \frac{\epsilon|P_0(j\omega)|}{\sqrt{1+|P_0(j\omega)|^2}\sqrt{1+(1-\epsilon)^2|P_0(j\omega)|^2}} \\
&= \frac{|P_0(j\omega) - (1-\epsilon)P_0(j\omega)|}{\sqrt{1+|P_0(j\omega)|^2}\sqrt{1+(1-\epsilon)^2|P_0(j\omega)|^2}} \\
&= \kappa(P_0(j\omega), (1-\epsilon)P_0(j\omega))
\end{aligned}$$

using (6.2); and consequently, for all $\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1$,

$$\begin{aligned}
\delta_\nu(P_0, (1+\epsilon\delta)P_0) &= \sup_\omega \kappa(P_0(j\omega), (1+\epsilon\delta(j\omega))P_0(j\omega)) \\
&\leq \sup_\omega \kappa(P_0(j\omega), (1-\epsilon)P_0(j\omega)) \\
&= \delta_\nu(P_0, (1-\epsilon)P_0).
\end{aligned}$$

[2] We say that $P_0 = NM^{-1}$, where $M, N \in \mathcal{RH}_\infty$, is a normalized (right) coprime factorization if $M^\sim M + N^\sim N = I$. Then, define $G := \left(\begin{smallmatrix} N \\ M \end{smallmatrix}\right)$. (See [86, 104] for more details.)

So

$$\sup_{\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1} \delta_\nu(P_0, (1 + \epsilon\delta)P_0) \leq \delta_\nu(P_0, (1 - \epsilon)P_0).$$

Therefore, the LHS of (6.3) is upper and lower bounded by the same quantity, and so, for a given $\epsilon \in [0, 1)$,

$$\sup_{\delta \in \mathcal{RH}_\infty : \|\delta\|_\infty \leq 1} \delta_\nu(P_0, (1 + \epsilon\delta)P_0) = \delta_\nu(P_0, (1 - \epsilon)P_0).$$

□

Given that we are considering SISO systems, and the case where $F_u(F, \delta) = (1 + \epsilon\delta)P_0$, $\epsilon \in [0, 1)$, notice that (5.3) and (6.1) are almost identical; the difference between the two conditions being determined by the closure of the LTI uncertainty set.

Combining Theorem 9 and (6.1) now gives us the following result.

Lemma 6. *Let $\epsilon \in [0, 1)$. If $[P_0, K]$ is internally stable and $\delta_\nu(P_0, (1-\epsilon)P_0) < b_{P_0,K}$, then the interconnection $[(1 + \epsilon\delta)P_0, K]$ is internally stable for all $\delta \in \mathcal{RH}_\infty$ such that $\|\delta\|_\infty \leq 1$.*

Proof. Follows directly from Theorem 9 and [90, Theorem 4.2]. □

6.4 The Stability Robustness Condition

The following theorem ties together results from the preceding sections in order to present a stability robustness condition for SISO systems subject to SISO, LTV, output-multiplicative uncertainty. The LHS of the stability robustness condition is analytically computable as only a standard, SISO, LTI ν-gap metric calculation is required.

Theorem 10. *Suppose $P_0 \in \mathcal{R}^{n\times m}$ is a nominal LTI plant and $K \in \mathcal{R}^{m\times n}$ is a controller such that $[P_0, K]$ is stable. Let $\epsilon \in [0, 1)$. If*

$$\delta_\nu(P_0, (1 - \epsilon)P_0) < b_{P_0,K},$$

then the LTV closed-loop system $[(1+\Delta_\epsilon)P_0, K]$ is stable for all Δ_ϵ such that $\|\Delta_\epsilon\| \leq \epsilon$.

Proof. By Lemma 6, it is known that if the LTI ν-gap metric between P_0 and $(1-\epsilon)P_0$ is sufficiently small, then the LTI closed-loop system $[(1 + \epsilon\delta)P_0, K]$ is stable for all $\delta \in \mathcal{RH}_\infty$ such that $\|\delta\|_\infty \leq 1$. Then, via Lemma 5, the LTV closed-loop system $[(1 + \Delta_\epsilon)P_0, K]$ must be stable for all Δ_ϵ such that $\|\Delta_\epsilon\| \leq \epsilon$, where $\Delta_\epsilon = \epsilon\Delta$. □

The following simple example illustrates the use of Theorem 10. Consider a nominal system $P_0 = \frac{1}{s+1}$. Calculation of the LHS of the stability robustness condition gives $\delta_\nu(P_0, (1-\epsilon)P_0) = \frac{\epsilon}{\sqrt{2}\sqrt{\epsilon^2-2\epsilon+2}}$ for $\epsilon \in [0, 1)$. This function is shown in Fig. 6.7.

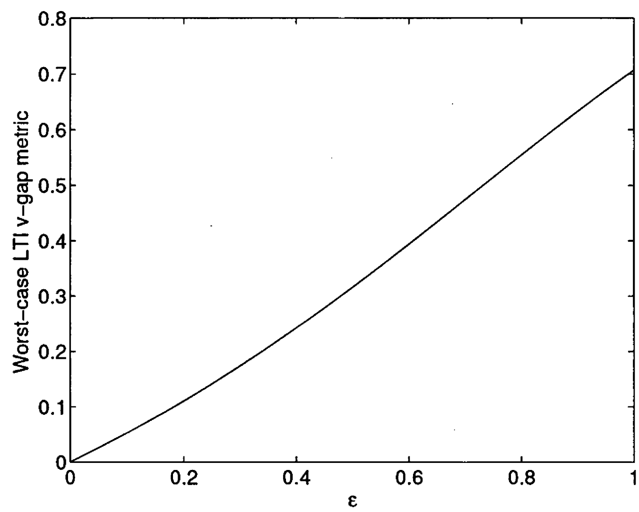

Fig. 6.7: Example.

Next, one would compare $\frac{\epsilon}{\sqrt{2}\sqrt{\epsilon^2-2\epsilon+2}}$ to $b_{P_0,K}$, given that K is some nominally stabilizing controller, in order to determine stability robustness of the closed-loop system. For instance, consider $\epsilon = 0.4$, which means that $\frac{\epsilon}{\sqrt{2}\sqrt{\epsilon^2-2\epsilon+2}} = \frac{1}{\sqrt{17}}$. In this case, all controllers achieving a generalized robust stability margin $b_{P_0,K}$ of greater than $\frac{1}{\sqrt{17}}$, are guaranteed to stabilize P_0 subject to causal, stable, LTV, output-multiplicative uncertainty of size less than or equal to 0.4. Furthermore, the second property of Lemma 1 guarantees that there exists a controller achieving a generalized robust stability margin $b_{P_0,K}$ of less than or equal to $\frac{1}{\sqrt{17}}$, that destabilizes P_0 subject to LTV output-multiplicative uncertainty of size less than or equal to 0.4.

6.5 Conclusions

A reduced version of the scaled LTI ν-gap metric condition for stability robustness, introduced in Chapter 5, was obtained for the case where SISO systems and LTV output-multiplicative uncertainty was considered. The part of the condition that formally required solution of a LMI feasibility problem (see Chapter 5), was then shown to be equivalent to a standard (and additionally, an analytically computable) LTI ν-gap metric quantity.

7. CONCLUDING COMMENTS

The overall aim of this dissertation was to present a number of stability results for feedback system interconnections. Below, the main results are summarized and possible future directions for research are outlined.

7.1 Conclusions

The following is a summary of the contributions presented in this dissertation.

A "mixed" small gain and passivity frequency domain property, based on the notion of dissipativity, was introduced in Chapter 3. A standard feedback interconnection consisting of two LTI systems, each with a "mixed" small gain and passivity frequency domain property associated with it, was then shown to be always input-output stable. The result paved the way for a similar result in regards to nonlinear systems, as follows.

In Chapter 4, a "mixed" small gain and passivity property was defined, in the time domain, for causal nonlinear systems. Systems exhibiting the strict form of this property were shown to automatically have finite gain. In the case where all systems involved were additionally stable and LTI, systems satisfying the condition for "mixed" small gain and passivity presented in this chapter, were shown to also satisfy the condition for "mixed" small gain and passivity defined in Chapter 3. The feedback interconnection consisting of two causal, nonlinear systems, each with a "mixed" small gain and passivity time domain property associated with it, was shown to be always input-output stable.

In Chapter 5, a standard, necessary and sufficient, scaled small gain condition for robust stability was extended into a (sufficient) scaled LTI ν-gap metric framework. Structured LTV uncertainty, and LTI nominal plants and controllers, were considered. The advantage of extending the scaled small gain condition into the ν-gap metric framework was that the LMI feasibility problem associated with checking the scaled LTI ν-gap metric condition was shown to be independent of K; once obtained, the scaled LTI ν-gap metric is compared to the generalized robust stability margin $b_{P_0,K}$. Thus the only role played by the controller K in the algorithm is in its effect on $b_{P_0,K}$, which must be known. On the other hand, the LMI feasibility problem associated with the scaled small gain stability robustness condition is dependent on knowledge of K at all frequencies.

In the case of SISO systems, and output-multiplicative LTV uncertainty, a reduced version of the scaled LTI ν-gap metric stability robustness condition may be obtained. This was done in Chapter 6. It was furthermore shown that (in this case) the scaled LTI ν-gap metric is equivalent to a standard, and additionally, an analytically computable, LTI ν-gap metric quantity.

7.2 Future Directions

Some possible directions for future research are as follows.

Using an appropriate choice of operators, dissipative systems concepts capture notions of small gain; and alternatively, passivity [40, 42, 44, 45, 70]. (Additionally, it is known that IQC concepts, where one of the systems in the feedback interconnection is required to be LTI, also capture the notions of small gain and passivity.) The "mixed" small gain and passivity frequency domain property was defined for LTI systems using the notion of dissipativity in Chapter 3. It would be of interest to examine the relationship between the "mixed" small gain and passivity property for nonlinear systems, described in Chapter 4, and dissipative systems notions. The paper [45] presents a technique for generating Lyapunov functions for systems with dissipative properties. It would be of interest to relate the notion of the nonlinear "mixed" small gain and passivity property to Lyapunov stability concepts as well.

The paper [47] generalized the Kalman-Yakubovich-Popov (KYP) lemma to establish a relationship between a frequency domain inequality in a finite frequency range, and a linear matrix inequality (LMI) condition. (The standard KYP lemma treats frequency domain inequalities, which characterize various properties of dynamical systems, for the entire frequency range only.) It was shown in [47, Section VI] that small gain and positive-real properties (amongst other properties) can be treated within the framework of this generalized KYP lemma. A natural extension would be to determine how the "mixed" small gain and passivity frequency domain property can be treated with the framework of the generalized KYP lemma.

The degree of difficulty in determining whether a nonlinear system satisfies the "mixed" small gain and passivity property, introduced in Chapter 4, has not yet been investigated.

It was shown that stable LTI systems satisfying the condition for "mixed" small gain and passivity defined in Chapter 4, also satisfied the condition for "mixed" small gain and passivity defined in Chapter 3. It would be ideal if this relationship between the conditions was necessary and sufficient. This may require the derivation of an alternative "nonlinear" condition.

Finally, any level of conservatism introduced into the stability robustness test described in Chapter 5, as a result of extending the scaled small gain condition into a scaled LTI ν-gap metric framework, should be further investigated.

APPENDIX

A. THE INDUCED REALIZATION

The aim of Appendix A is to formally define the induced realization for an upper or lower linear fractional transformation. Results concerning the stabilizability and detectability of induced realizations are provided, and internal stability is thoroughly described in the sense of the induced realizations.

A.1 Induced Realizations for LFTs

Let a stabilizable and detectable realization[1] for $F \in \mathcal{R}^{(p+n)\times(q+m)}$, a generalized system, be given by

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right), \tag{A.1}$$

and let stabilizable and detectable realizations for a controller $K \in \mathcal{R}^{m\times n}$ and an uncertain system $E \in \mathcal{R}^{q\times p}$ be given by $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$, respectively. Note that a realization (A, B, C, D) for a transfer function matrix $X(s)$ is stabilizable and detectable if and only if $\bar{n}(X) = \bar{\lambda}(A)$, where $\bar{n}(\cdot)$ denotes the number of closed RHP poles counted according to the usual notion of the Smith-McMillan decomposition and $\bar{\lambda}(\cdot)$ denotes the number of eigenvalues with real part in the closed RHP. This result is a consequence of the fact that the only uncontrollable and unobservable modes in (A, B, C, D) must be in $Re(s) < 0$ if the realization is stabilizable and detectable.

Definition 11. *The induced realization for*

(a) $F_l(F, K)$ is the realization formed from the above-stated realizations for $F \in \mathcal{R}^{(p+n)\times(q+m)}$ and $K \in \mathcal{R}^{m\times n}$ as given by

$$\left(\begin{array}{c|c} A_\theta & B_\theta \\ \hline C_\theta & D_\theta \end{array}\right),$$

[1] The D_{22} term has been absorbed into the controller by a loop shifting argument (see [32, Section 4.6] for instance).

where

$$
\begin{aligned}
A_\theta &:= \begin{pmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{pmatrix} \\
B_\theta &:= \begin{pmatrix} B_1 + B_2\hat{D}D_{21} \\ \hat{B}D_{21} \end{pmatrix} \\
C_\theta &:= \begin{pmatrix} C_1 + D_{12}\hat{D}C_2 & D_{12}\hat{C} \end{pmatrix} \\
D_\theta &:= D_{11} + D_{12}\hat{D}D_{21};
\end{aligned}
$$

(b) $F_u(F,E)$ *is the realization formed from the above-stated realizations for* $F \in \mathcal{R}^{(p+n)\times(q+m)}$ *and* $E \in \mathcal{R}^{q\times p}$ *as given by*

$$
\left(\begin{array}{c|c} A_\eta & B_\eta \\ \hline C_\eta & D_\eta \end{array}\right), \tag{A.2}
$$

where

$$
\begin{aligned}
A_\eta &:= \begin{pmatrix} \breve{A} + \breve{B}RD_{11}\breve{C} & \breve{B}RC_1 \\ B_1Q\breve{C} & A + B_1Q\breve{D}C_1 \end{pmatrix} \\
B_\eta &:= \begin{pmatrix} \breve{B}RD_{12} \\ B_1Q\breve{D}D_{12} + B_2 \end{pmatrix} \\
C_\eta &:= \begin{pmatrix} D_{21}Q\breve{C} & D_{21}Q\breve{D}C_1 + C_2 \end{pmatrix} \\
D_\eta &:= D_{21}Q\breve{D}D_{12}
\end{aligned}
$$

and $R := (I - D_{11}\breve{D})^{-1}$, $Q := (I - \breve{D}D_{11})^{-1}$.

Stabilizability and detectability of the induced realization for $F_u(F,E)$ is considered in the next result.

Theorem 11. *Consider a generalized plant* $F \in \mathcal{R}^{(p+n)\times(q+m)}$ *and a LTI uncertainty* $E \in \mathcal{R}^{q\times p}$. *Suppose that a stabilizable and detectable realization for* F *is given by* (A.1) *and that* E *has a given stabilizable and detectable realization. Then the induced realization for* $F_u(F,E)$, *defined as in Definition 11, is stabilizable and detectable if and only if* $\bar{n}(F_u(F,E)) = \bar{n}(T)$, *where* $\bar{n}(\cdot)$ *denotes the number of closed RHP poles counted according to the usual notion of the Smith-McMillan decomposition, and* $T(s)$ *denotes the transfer function matrix mapping* $(d_2'\ d_1'\ u')'$ *to* $(w'\ z'\ y')'$ *as shown in Fig. A.1. Also, the induced realization for* $F_u(F,E)$ *is*

(a) detectable if $\begin{pmatrix} A - \lambda I & B_1 \\ C_2 & D_{21} \end{pmatrix}$ *has full column rank* $\forall Re(\lambda) \geq 0$;

(b) stabilizable if $\begin{pmatrix} A-\lambda I & B_2 \\ C_1 & D_{12} \end{pmatrix}$ *has full row rank* $\forall Re(\lambda) \geq 0$.

Proof. Parts (a) and (b) of the final part of Theorem 11 are from [104, Lemma 16.1], with their proof given in that reference. It remains to prove the necessary and sufficient condition for stabilizability and detectability of the induced realization.

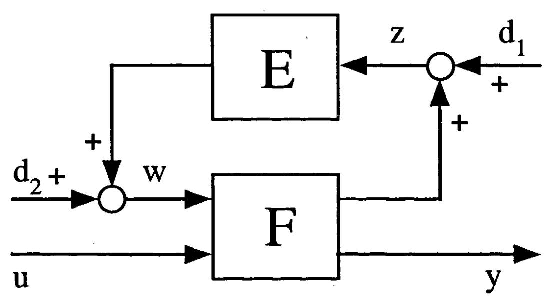

Fig. A.1: Mapping of $(d_2' \; d_1' \; u')'$ to $(w' \; z' \; y')'$.

Let the stabilizable and detectable realization for E be given by $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$. The induced realization for $T(s)$ is given by

$$\left(\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right), \tag{A.3}$$

where

$$A_T := \begin{pmatrix} \breve{A} + \breve{B}RD_{11}\breve{C} & \breve{B}RC_1 \\ B_1Q\breve{C} & A + B_1Q\breve{D}C_1 \end{pmatrix}$$
$$B_T := \begin{pmatrix} \breve{B}RD_{11} & \breve{B}R & \breve{B}RD_{12} \\ B_1Q & B_1Q\breve{D} & B_1Q\breve{D}D_{12} + B_2 \end{pmatrix}$$
$$C_T := \begin{pmatrix} Q\breve{C} & Q\breve{D}C_1 \\ RD_{11}\breve{C} & RC_1 \\ D_{21}Q\breve{C} & D_{21}Q\breve{D}C_1 + C_2 \end{pmatrix}$$
$$D_T := \begin{pmatrix} Q & Q\breve{D} & Q\breve{D}D_{12} \\ RD_{11} & R & RD_{12} \\ D_{21}Q & D_{21}Q\breve{D} & D_{21}Q\breve{D}D_{12} \end{pmatrix}$$

and $R := (I - D_{11}\breve{D})^{-1}$, $Q := (I - \breve{D}D_{11})^{-1}$. This induced realization is stabilizable and detectable. To see detectability, suppose that

$$\begin{pmatrix} \breve{A}+\breve{B}RD_{11}\breve{C}-\lambda I & \breve{B}RC_1 \\ B_1Q\breve{C} & A+B_1Q\breve{D}C_1-\lambda I \\ Q\breve{C} & Q\breve{D}C_1 \\ RD_{11}\breve{C} & RC_1 \\ D_{21}Q\breve{C} & D_{21}Q\breve{D}C_1+C_2 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, row 3 and row 4 $\Rightarrow C_1w_2 = 0$ and $\breve{C}w_1 = 0$; row 5 $\Rightarrow C_2w_2 = 0$, row 1 $\Rightarrow (\breve{A}-\lambda I)w_1 = 0$ and row 2 $\Rightarrow (A-\lambda I)w_2 = 0$. Since $(\left(\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right), A)$ and $(\breve{C}, \breve{A})$ are detectable, $w_1 = 0$ and $w_2 = 0$ for all $Re(\lambda) \geq 0$ and so (A.3) is detectable. Stabilizability of (A.3) may be similarly established. So $\bar{n}(T) = \bar{\lambda}(A_T)$.

But from observation of (A.2), $A_T = A_\eta$. Consequently, it must be shown that (A.2) is stabilizable and detectable if and only if $\bar{n}(F_u(F,E)) = \bar{\lambda}(A_\eta)$. If (A.2) is stabilizable and detectable, then $\bar{n}(F_u(F,E)) = \bar{\lambda}(A_\eta)$. If (A.2) is not stabilizable and/or not detectable, then A_η has an unstable hidden mode which implies that $\bar{n}(F_u(F,E)) < \bar{\lambda}(A_\eta)$. □

Parts (a) and (b) of the final part of Theorem 11 have been given because the necessary and sufficient condition stated in the earlier part of the theorem is dependent on E, and can hence be difficult to check. This is as opposed to the sufficient conditions in (a) and (b), which are equivalent to requiring no unstable invariant zeros of the realizations

$$\left(\begin{array}{c|c} A & B_1 \\ \hline C_2 & D_{21} \end{array}\right) \text{ and } \left(\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right),$$

for F_{21} and F_{12}, respectively.

Now suppose that $P_0 := F_u(F,0) = F_{22}$ has a realization (A, B_2, C_2) which is inherited from (A.1), and suppose further that this realization is stabilizable and detectable (see Footnote 2 in Section 5.3). Then suppose that K has a stabilizable and detectable realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, and that $[P_0, K]$ as shown in Fig. 5.1 is internally stable. An immediate consequence is stabilizability and detectability of the induced realization for $F_l(F, K)$. To see this, suppose that x and $\hat{x}$ denote the state vectors for the realizations for P_0 and K, respectively; the state equations corresponding to Fig. 5.1 with $v_1 = v_2 = 0$ are

$$\dot{x} = Ax + B_2u \tag{A.4}$$

$$y = C_2x \tag{A.5}$$

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}y \tag{A.6}$$

$$u = \hat{C}\hat{x} + \hat{D}y. \tag{A.7}$$

Solving (A.5) and (A.7) for u and y, and substituting into (A.4) and (A.6) gives

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \tilde{A} \begin{pmatrix} x \\ \hat{x} \end{pmatrix},$$

where $\tilde{A} := \begin{pmatrix} A+B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{pmatrix}$ is Hurwitz [104, Lemma 5.2]. But $\tilde{A} = A_\theta$, where A_θ is as given in Definition 11. So the induced realization for $F_l(F, K)$ is stabilizable and detectable. In fact, since A_θ is Hurwitz (meaning that the system in Fig. A.2 is internally stable), we also know that $F_l(F, K) \in \mathcal{RH}_\infty$ (see [104, Lemma 12.2]).

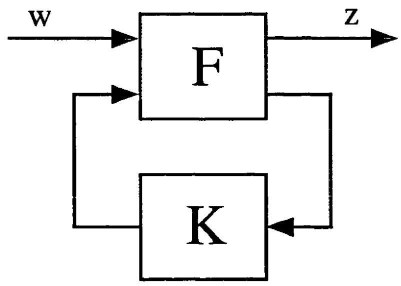

Fig. A.2: System representation of $F_l(F, K)$.

A.2 Internal Stability

Considering the induced realizations defined above, internal stability is now described.

Lemma 7. *Consider a generalized plant $F \in \mathcal{R}^{(p+n)\times(q+m)}$, a controller $K \in \mathcal{R}^{m\times n}$ and a LTI uncertain system $E \in \mathcal{R}^{q\times p}$. Let induced realizations for $F_l(F, K)$ and $F_u(F, E)$ be as defined in Definition 11, and suppose that these induced realizations are stabilizable and detectable. Then the interconnection $[E, F_l(F, K)]$ as shown in Fig. A.3 is internally stable if and only if the interconnection $[F_u(F, E), K]$ as shown in Fig. A.4 is internally stable; that is, the point of external signal injection, whether it is between between the uncertainty E and the system $F_l(F, K)$ (as shown in Fig. A.3) or between the system $F_u(F, E)$ and the controller K (as shown in Fig. A.4), is irrelevant in the definition of internal stability.*

Proof. Let stabilizable and detectable realizations for K and E be $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$, respectively. Consider the system shown in Fig. A.3. Letting $\breve{x}$ and x_θ denote the state vectors for the realization for E and the induced realization for

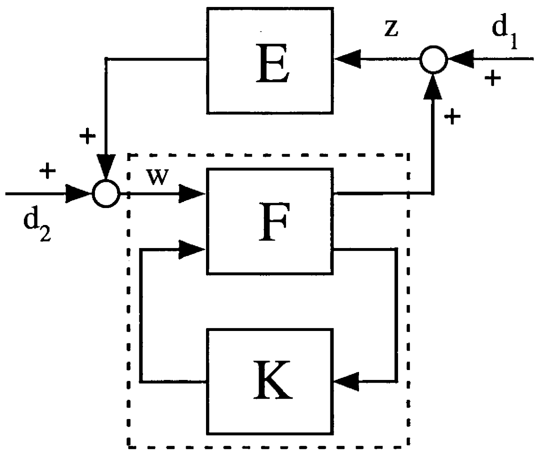

Fig. A.3: Internal stability of $[E, F_l(F, K)]$.

$F_l(F, K)$, respectively, and writing the state equations for the system with $d_1 = d_2 = 0$ gives

$$\dot{\breve{x}} = \breve{A}\breve{x} + \breve{B}z \tag{A.8}$$

$$w = \breve{C}\breve{x} + \breve{D}z \tag{A.9}$$

$$\dot{x}_\theta = A_\theta x_\theta + B_\theta w \tag{A.10}$$

$$z = C_\theta x_\theta + D_\theta w \tag{A.11}$$

where $(A_\theta, B_\theta, C_\theta, D_\theta)$ denotes the induced realization for $F_l(F, K)$. Solving (A.9) and (A.11) for z and w, and substituting into (A.8) and (A.10), gives

$$\begin{pmatrix} \dot{\breve{x}} \\ \dot{x}_\theta \end{pmatrix} = \mathcal{A} \begin{pmatrix} \breve{x} \\ x_\theta \end{pmatrix}, \tag{A.12}$$

where $\mathcal{A} := \begin{pmatrix} \breve{A} & 0 \\ 0 & A_\theta \end{pmatrix} + \begin{pmatrix} \breve{B} & 0 \\ 0 & B_\theta \end{pmatrix} \begin{pmatrix} I & -D_\theta \\ -\breve{D} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & C_\theta \\ \breve{C} & 0 \end{pmatrix}$. By similarly considering the system shown in Fig. A.4 and letting x_η and $\hat{x}$ denote the state vectors for the induced realization for $F_u(F, E)$ and the realization for K, respectively, rearrangement of the corresponding state equations with $v_1 = v_2 = 0$ gives

$$\begin{pmatrix} \dot{x}_\eta \\ \dot{\hat{x}} \end{pmatrix} = \mathbb{A} \begin{pmatrix} x_\eta \\ \hat{x} \end{pmatrix}, \tag{A.13}$$

where $\mathbb{A} := \begin{pmatrix} A_\eta & 0 \\ 0 & \hat{A} \end{pmatrix} + \begin{pmatrix} B_\eta & 0 \\ 0 & \hat{B} \end{pmatrix} \begin{pmatrix} I & -\hat{D} \\ -D_\eta & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & \hat{C} \\ C_\eta & 0 \end{pmatrix}$ and $(A_\eta, B_\eta, C_\eta, D_\eta)$ denotes the induced realization for $F_u(F, E)$. But a simple calculation shows that $\mathcal{A} = \mathbb{A}$.

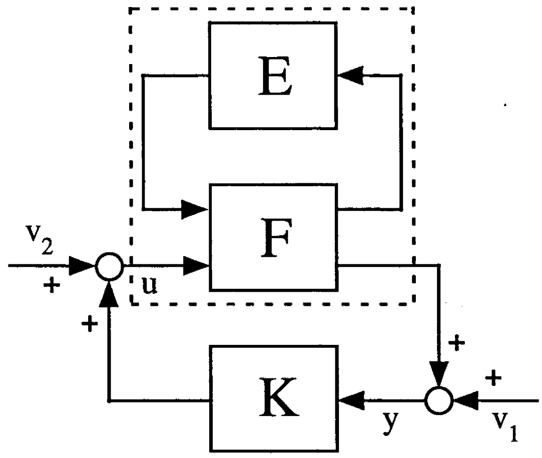

Fig. A.4: Internal stability of $[F_u(F, E), K]$.

From [104, Lemma 5.2], $[E, F_l(F, K)]$ is internally stable if and only if $\mathcal{A}$ is Hurwitz; and $[F_u(F, E), K]$ is internally stable if and only if $\mathbb{A}$ is Hurwitz. So $[E, F_l(F, K)]$ is internally stable if and only if $[F_u(F, E), K]$ is internally stable. □

A.3 Linear Time-Varying Internal Stability

As expected, Lemma 7 is easily extended to provide for the case of when the LTI uncertainty, E, is replaced by a causal LTV uncertainty, E_{LTV}. Let E_{LTV} be described by the state-space equations

$$\dot{\breve{x}} = \breve{A}(t)\breve{x} + \breve{B}(t)z, \;\; \breve{x}(0) = \breve{x}_0, \tag{A.14}$$

$$w = \breve{C}(t)\breve{x} + \breve{D}(t)z \tag{A.15}$$

where $t \in \mathbb{R}_+$, $z(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^q$ and $\breve{A}(t)$, $\breve{B}(t)$, $\breve{C}(t)$ and $\breve{D}(t)$ are bounded functions of time. Let $\Phi(t, \tau)$ be the transition matrix associated with (A.14) and (A.15), which is the solution to the array of first-order differential equations

$$\frac{d}{dt}\Phi(t, \tau) = \breve{A}(t)\Phi(t, \tau), \;\; \Phi(\tau, \tau) = I.$$

Definition 12. *[76] A system as described by* (A.14) *and* (A.15) *is said to be exponentially stable if there exist* $c_1, c_2 > 0$ *such that, for all* $t \geq \tau$,

$$\|\Phi(t, \tau)\| \leq c_1 e^{-c_2(t-\tau)},$$

where $t, \tau \in \mathbb{R}_+$.

Definition 13. *[76] A system as described by* (A.14) *and* (A.15) *is said to admit a stabilizable (respectively, detectable) realization* $(\breve{A}(t), \breve{B}(t))$ *(respectively,* $(\breve{C}(t), \breve{A}(t))$*) if there exists a bounded matrix function of time* $\breve{F}(t)$ *(respectively,* $\breve{L}(t)$*) such that the system* $\dot{\breve{x}} = (\breve{A} + \breve{B}\breve{F})(t)\breve{x}$ *(respectively,* $\dot{\breve{x}} = (\breve{A} + \breve{L}\breve{C})(t)\breve{x}$*) is exponentially stable.*[2]

Lemma 8. *Consider a generalized plant* $F \in \mathcal{R}^{(p+n)\times(q+m)}$*, a controller* $K \in \mathcal{R}^{m\times n}$ *and a causal LTV uncertainty* E_{LTV} *that is described by the equations* (A.14) *and* (A.15)*. Let induced realizations for* $F_l(F, K)$ *and* $F_u(F, E_{LTV})$ *be as defined in Definition 11, with (a stabilizable and detectable realization for)* E_{LTV} *replacing (a stabilizable and detectable realization for)* $E \in \mathcal{R}^{q\times p}$*. Suppose that these induced realizations are stabilizable and detectable. Then the interconnection* $[E_{LTV}, F_l(F, K)]$ *as shown in Fig. A.5 is internally stable if and only if the interconnection* $[F_u(F, E_{LTV}), K]$ *as shown in Fig. A.6 is internally stable.*

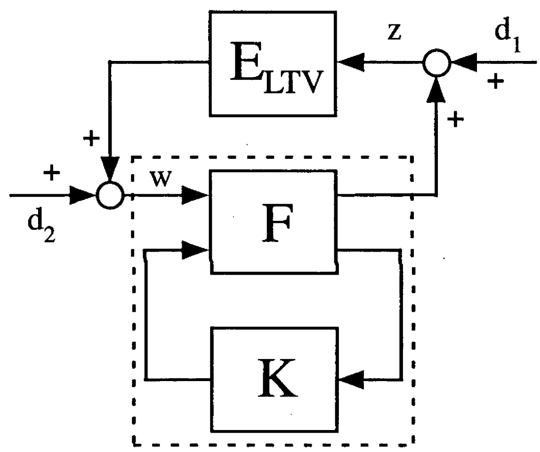

Fig. A.5: Internal stability of $[E_{LTV}, F_l(F, K)]$.

Proof. The proof follows in the spirit of Lemma 7's proof. That is, constructions of $\mathcal{A}$ and $\mathbb{A}$ are straightforward in the LTV uncertainty case; and again a simple calculation shows that $\mathcal{A} = \mathbb{A}$. Then it is noted that exponential stability of systems (A.12) and (A.13) is necessary and sufficient for internal stability of the interconnections $[E_{LTV}, F_l(F, K)]$ and $[F_u(F, E_{LTV}), K]$, respectively. □

[2] It should be noted that the concepts of "stabilizability" and "detectability" as defined in [76] are traditionally referred to as uniform stabilizability and uniform detectability (see [2] for alternative, discrete-time notions of uniform stabilizability and uniform detectability).

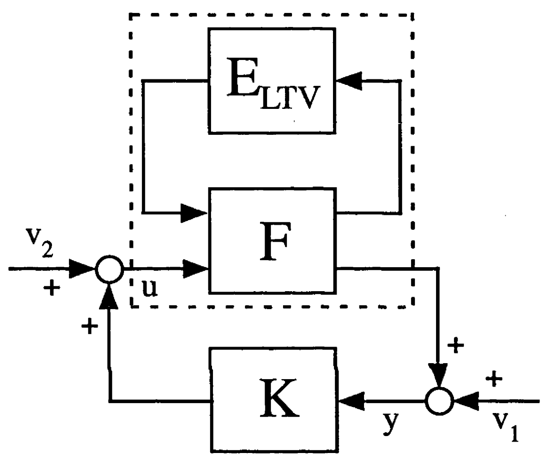

Fig. A.6: Internal stability of $[F_u(F, E_{LTV}), K]$.

B. $A_{F_L} + B_{F_L} F_{F_L}$ IS HURWITZ

The proof of Lemma 4 requires us to show that the matrix denoted $A_{F_l} + B_{F_l} F_{F_l}$ is Hurwitz. First, a review of the chain-scattering representation of a system, particularly $\tilde{R}$ as defined in (5.6), is required. Recall the input-output representation of $\tilde{R}$:

$$\begin{pmatrix} z \\ a_1 \end{pmatrix} = \tilde{R} \begin{pmatrix} w \\ a_2 \end{pmatrix}, \tag{B.1}$$

as shown in Fig. B.1. Since $(\tilde{R}_{21})^{-1}$ exists and is proper, (B.1) can be alternatively represented as

$$\begin{pmatrix} z \\ w \end{pmatrix} = CHAIN(\tilde{R}) \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}, \tag{B.2}$$

where

$$CHAIN(\tilde{R}) := \begin{pmatrix} \tilde{R}_{12} - \tilde{R}_{11}(\tilde{R}_{21})^{-1}\tilde{R}_{22} & \tilde{R}_{11}(\tilde{R}_{21})^{-1} \\ -(\tilde{R}_{21})^{-1}\tilde{R}_{22} & (\tilde{R}_{21})^{-1} \end{pmatrix}.$$

Relation (B.2) is referred to as the chain-scattering representation of $\tilde{R}$, as shown in Fig. B.2. A state-space representation for $CHAIN(\tilde{R})$ is

$$\left(\begin{array}{c|cc} A_{\tilde{R}} - \frac{1}{\sqrt{\gamma^2-1}} B_{\tilde{R}_1} C_{\tilde{R}_2} & B_{\tilde{R}_2} & \frac{1}{\sqrt{\gamma^2-1}} B_{\tilde{R}_1} \\ \hline C_{\tilde{R}_1} & I & 0 \\ -\frac{1}{\sqrt{\gamma^2-1}} C_{\tilde{R}_2} & 0 & \frac{1}{\sqrt{\gamma^2-1}} I \end{array}\right) \tag{B.3}$$

(refer to [60, Chapter 4.2] for the general state-space formula for $CHAIN(\cdot)$).

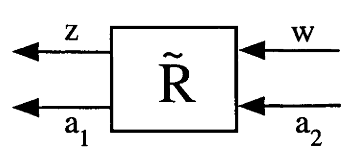

Fig. B.1: Input-output representation of $\tilde{R}$.

Fig. B.2: Chain-scattering representation of $\tilde{R}$.

Now $CHAIN(\tilde{R}) \in \mathcal{RH}_\infty$ since

$$CHAIN(\tilde{R}) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} CHAIN(R) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and $CHAIN(R) \in \mathcal{RH}_\infty$ [13]. Furthermore, (B.3) is stabilizable and detectable since there exist matrices $\bar{F} := \begin{pmatrix} -C_{\tilde{R}_1} \\ C_{\tilde{R}_2} \end{pmatrix}$ and $\bar{L} := \begin{pmatrix} -B_{\tilde{R}_2} & -B_{\tilde{R}_1} \end{pmatrix}$ such that $\bar{A} + \begin{pmatrix} B_{\tilde{R}_2} & \frac{1}{\sqrt{\gamma^2-1}} B_{\tilde{R}_1} \end{pmatrix} \bar{F}$ and $\bar{A} + \bar{L} \begin{pmatrix} C_{\tilde{R}_1} \\ -\frac{1}{\sqrt{\gamma^2-1}} C_{\tilde{R}_2} \end{pmatrix}$, respectively, are Hurwitz, where $\bar{A} := A_{\tilde{R}} - \frac{1}{\sqrt{\gamma^2-1}} B_{\tilde{R}_1} C_{\tilde{R}_2}$ (see Footnote 3 of Chapter 5). So $\bar{A}$ is Hurwitz. But

$$A_{F_l} + B_{F_l} F_{F_l} = \begin{pmatrix} \bar{A} & \bullet \\ 0 & A_{P_1} + B_{P_1} F_{P_1} \end{pmatrix},$$

where $\bullet$ denotes a "don't care" element, so $A_{F_l} + B_{F_l} F_{F_l}$ is Hurwitz.

BIBLIOGRAPHY

[1] B. D. O. Anderson, "Failures of adaptive control theory and their resolution". *Communications in Information and Systems*, vol. 5, no. 1, pp. 1-20, 2005.

[2] B. D. O. Anderson and J. B. Moore, "Detectability and stabilizability of time-varying discrete-time linear systems". *SIAM Journal on Control and Optimization*, vol. 19, no. 1, pp. 21-32, 1981.

[3] B. D. O. Anderson, T. S. Brinsmead and F. De Bruyne, "The Vinnicombe metric for nonlinear operators". *IEEE Transactions on Automatic Control*, vol. 47, no. 9, pp. 1450-1465, 2002.

[4] B. D. O. Anderson, R. R. Bitmead, C. R. Johnson Jr., P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly and B. D. Riedle, *Stability of adaptive systems: passivity and averaging analysis.* The MIT Press, Cambridge, Massachusetts, 1986.

[5] B. D. O. Anderson and S. Vongpanitlerd, *Network analysis and synthesis.* Prentice Hall, Englewood Cliffs, New Jersey, 1973.

[6] M. Araki and M. Saeki, "A quantitative condition for the well-posedness of interconnected dynamical systems". *IEEE Transactions on Automatic Control*, vol. 28, no. 5, pp. 569-577, 1983.

[7] R. A. Baker and A. R. Bergen, "Lyapunov stability and Lyapunov functions of infinite dimensional systems". *IEEE Transactions on Automatic Control*, vol. 14, no. 4, pp. 325-334, 1969.

[8] S. Boyd and Q. Yang, "Structured and simultaneous Lyapunov functions for system stability problems". *International Journal of Control*, vol. 49, no. 6, pp. 2215-2240, 1989.

[9] R. D. Braatz and M. Morari, "On the stability of systems with mixed time-varying parameters". *International Journal of Robust and Nonlinear Control*, vol. 7, pp. 105-112, 1997.

[10] F. M. Callier and C. A. Desoer, "Simplifications and clarifications on the paper "An algebra of transfer functions for distributed linear time-invariant systems"". *IEEE Transactions on Circuits and Systems*, vol. 27, no. 4, pp. 320-323, 1980.

[11] F. M. Callier and C. A. Desoer, "An algebra of transfer functions for distributed linear time-invariant systems". *IEEE Transactions on Circuits and Systems*, vol. 25, no. 9, pp. 651-662, 1978.

[12] M. Cantoni, "On model reduction in the ν-gap metric". *Proceedings of the 40th IEEE Conference on Decision and Control*, pp. 3665-3670, Orlando, Florida, USA, 2001.

[13] M. Cantoni and G. Vinnicombe, "Controller discretisation: a gap metric framework for analysis and synthesis". *IEEE Transactions on Automatic Control*, vol. 49, no. 11, pp. 2033-2039, 2004.

[14] M. Cantoni and G. Vinnicombe, "Controller discretisation: a gap metric framework for analysis and synthesis". Technical report TR04-04, Department of Electrical and Electronic Engineering, The University of Melbourne, 2004.

[15] Y. S. Cho and K. S. Narendra, "Stability of nonlinear time-varying feedback systems". *Proceedings of the 5th Allerton Conference*, pp. 249-258, 1967.

[16] R. F. Curtain and H. J. Zwart, *An introduction to infinite-dimensional linear systems theory.* Springer-Verlag, New York, 1995.

[17] M. A. Dahleh and I. J. Diaz-Bobillo, *Control of uncertain systems: a linear programming approach.* Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[18] R. M. De Santis, "Causality theory in systems analysis". *Proceedings of the IEEE*, vol. 64, no. 1, pp. 36-44, 1976.

[19] C. A. Desoer and M. Vidyasagar, *Feedback systems: input-output properties.* Academic Press, New York, 1975.

[20] G. E. Dullerud and F. Paganini, *A course in robust control theory: a convex approach.* Springer-Verlag, New York, 2000.

[21] A. K. El-Sakkary, "The gap metric: robustness of stabilization of feedback systems". *IEEE Transactions on Automatic Control*, vol. 30, no. 3, pp. 240-247, 1985.

[22] R. F. Estrada and C. A. Desoer, "Passivity and stability with a state representation". *International Journal of Control*, vol. 13, pp. 1-26, 1971.

[23] A. Feintuch, "The gap metric for time-varying systems". *Systems & Control Letters*, vol. 16, pp. 277-279, 1991.

[24] M. Fukuda and M. Kojima, "Branch-and-cut algorithms for the bilinear matrix inequality eigenvalue problem". *Computational Optimization and Applications*, vol. 19, pp. 79-105, 2001.

[25] T. T. Georgiou, "Differential stability and robust control of nonlinear systems". *Mathematics of Control, Signals, and Systems*, vol. 6, pp. 289-306, 1993.

[26] T. T. Georgiou, "On the computation of the gap metric". *Systems & Control Letters*, vol. 11, pp. 253-257, 1988.

[27] T. T. Georgiou and M. C. Smith, "Robustness analysis of nonlinear feedback systems: an input-output approach". *IEEE Transactions on Automatic Control*, vol. 42, no. 9, pp. 1200-1221, 1997.

[28] T. T. Georgiou and M. C. Smith, "Optimal robustness in the gap metric". *IEEE Transactions on Automatic Control*, vol. 35, no. 6, pp. 673-685, 1990.

[29] T. T. Georgiou, A. Pascoal and P. P. Khargonekar, "On the robust stabilizability of uncertain linear time-invariant plants using nonlinear time-varying controllers". *Automatica*, vol. 35, no. 5, pp. 617-624, 1987.

[30] K. Glover and D. McFarlane, "Robust stabilization of normalized coprime factor plant descriptions with $\mathcal{H}_\infty$-bounded uncertainty". *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 821-830, 1989.

[31] N. E. Gough and A. K. El-Sakkary, "Basic procedure for model comparison using the gap metric". *International Journal of Systems Science*, vol. 16, no. 5, pp. 589-604, 1985.

[32] M. Green and D. J. N. Limebeer, *Linear robust control.* Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[33] W. M. Griggs, B. D. O. Anderson, A. Lanzon and M. Rotkowitz, "Interconnections of nonlinear systems with "mixed" small gain and passivity properties and associated input-output stability results". Submitted to *Systems & Control Letters* on 18th June, 2007.

[34] W. M. Griggs, B. D. O. Anderson and A. Lanzon, "A "mixed" small gain and passivity theorem for an interconnection of linear time-invariant systems". Accepted for presentation at the *European Control Conference 2007*, Kos, Greece, July, 2007.

[35] W. M. Griggs, B. D. O. Anderson and A. Lanzon, "A "mixed" small gain and passivity theorem in the frequency domain". Accepted for publication in *Systems & Control Letters* on 2nd April, 2007.

[36] W. M. Griggs, B. D. O. Anderson, A. Lanzon and M. Rotkowitz, "A stability result for interconnections of nonlinear systems with "mixed" small gain and passivity properties". Submitted to *46th IEEE Conference on Decision and Control* on 8th March, 2007.

[37] W. M. Griggs, A. Lanzon and B. D. O. Anderson, "A stability robustness test for systems with linear time-varying uncertainties". *Proceedings of the 44th IEEE Conference on Decision and Control and the 2005 European Control Conference*, pp. 3651-3656, Seville, Spain, 2005.

[38] W. M. Griggs, A. Lanzon and B. D. O. Anderson, "On Vinnicombe metrics and the stability robustness of linear time-varying systems". *Proceedings of the 5th Asian Control Conference*, vol. 3, pp. 1469-1477, Melbourne, Australia, 2004.

[39] C. J. Harris and J. M. E. Valenca, *The stability of input-output dynamical systems.* Academic Press, London, UK, 1983.

[40] D. J. Hill and P. J. Moylan, "General instability results for interconnected systems". *SIAM Journal on Control and Optimization*, vol. 21, no. 2, pp. 256-279, 1983.

[41] D. J. Hill and P. J. Moylan, "Connections between finte-gain and asymptotic stability". *IEEE Transactions on Automatic Control*, vol. 25, no. 5, pp. 931-936, 1980.

[42] D. J. Hill and P. J. Moylan, "Dissipative dynamical systems: basic input-output and state properties". *Journal of the Franklin Institute*, vol. 309, no. 5, pp. 327-357, 1980.

[43] D. J. Hill and P. J. Moylan, "A general result on the instability of feedback systems". *IEEE International Symposium on Circuits and Systems*, New York, 1978.

[44] D. J. Hill and P. J. Moylan, "Stability results for nonlinear feedback systems". *Automatica*, vol. 13, pp. 377-382, 1977.

[45] D. Hill and P. Moylan, "The stability of nonlinear dissipative systems". *IEEE Transactions on Automatic Control*, vol. 21, no. 5, pp. 708-711, 1976.

[46] D. J. Hill and P. J. Moylan, "Cyclo-dissipativeness, dissipativenesss and losslessness for nonlinear dynamical systems". Department of Electrical Engineering, University of Newcastle, NSW, Australia, *Tech. Report No. EE-7526*, 1975.

[47] T. Iwasaki and S. Hara, "Genrealized KYP lemma: unified frequency domain inequalities with design applications". *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 41-59, 2005.

[48] T. Iwasaki, S. Hara and A. L. Fradkov, "Restricted frequency inequality is equivalent to restricted dissipativity". *Proceedings of the 43rd IEEE Conference on Decision and Control*, pp. 426-431, Atlantis, Paradise Island, Bahamas, 2004.

[49] T. Iwasaki, S. Hara and H. Yamauchi, "Dynamical system design from a control perspective: finite frequency positive-realness approach". *IEEE Transactions on Automatic Control*, vol. 48, no. 8, pp. 1337-1354, 2003.

[50] T. Iwasaki, S. Hara and H. Yamauchi, "Structure/control design integration with finite frequency positive real property". *Proceedings of the American Control Conference*, pp. 549-553, Chicago, Illinois, USA, 2000.

[51] C. A. Jacobson, A. M. Stanković and G. Tadmor, "On the reduction of certain frequency-shaped linear quadratic dissipative design problems to an $\mathcal{H}_\infty$ formulation". *IEEE Transactions on Automatic Control*, vol. 41, no. 1, pp. 121-125, 1996.

[52] M. R. James, M. C. Smith and G. Vinnicombe, "Gap metrics, representations, and nonlinear robust stability". *Proceedings of the 39th IEEE Conference on Decision and Control*, pp. 2936-2941, Sydney, Australia, 2000.

[53] U. Jönsson, *Lecture notes on integral quadratic constraints.* ISSN 1401-2294, 2001.

[54] U. Jönsson and A. Rantzer, "Systems with uncertain parameters - time-variations with bounded derivatives". *Proceedings of the 33rd IEEE Conference on Decision and Control*, vol. 3, pp. 3074-3079, Lake Buena Vista, Florida, USA, 1994.

[55] R. E. Kalman, "Lyapunov functions for the problem of Lur'e in automatic control". *Proceedings of the National Academy of Sciences of the United States of America*, vol. 49, no. 2, pp. 201-205, 1963.

[56] R. E. Kalman, "A new approach to linear filtering and prediction problems". *Transactions of the ASME (Journal of Basic Engineering)*, vol. 82D, pp. 35-45, 1960.

[57] R. E. Kalman, "Contributions to the theory of optimal control". *Boletin de la Sociedad Matematica Mexicana*, vol. 5, pp. 102-119, 1960.

[58] R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory". *Transactions of the ASME (Journal of Basic Engineering)*, vol. 83D, pp. 95-108, 1961.

[59] M. Khammash, M. V. Salapaka and T. Van Voorhis, "Robust synthesis in l_1: a globally optimal solution". *IEEE Transactions on Automatic Control*, vol. 46, no. 11, pp. 1744-1754, 2001.

[60] H. Kimura, *Chain-scattering approach to $\mathcal{H}^\infty$-control*. Birkhäuser Boston, Cambridge, Massachusetts, 1997.

[61] A. Lanzon, "Weight optimization in $\mathcal{H}_\infty$ loop-shaping". *Automatica*, vol. 41, pp. 1201-1208, 2005.

[62] A. M. Lyapunov, "The general problem of the stability of motion". *International Journal of Control*, vol. 55, pp. 531-773, 1992 (this is the English translation of the Russian original published in 1892).

[63] I. M. Y. Mareels, B. D. O. Anderson, R. R. Bitmead, M. Bodson and S. S. Sastry, "Revisiting the MIT rule for adaptive control". *Proceedings of the 2nd IFAC Workshop on Adaptive Systems in Control and Signal Processing*, pp. 161-166, Lund, Sweden, 1986.

[64] H. J. Marquez, *Nonlinear control systems: analysis and design.* John Wiley & Sons, Hoboken, New Jersey, 2003.

[65] D. McFarlane and K. Glover, "A loop shaping design procedure using $\mathcal{H}_\infty$ synthesis". *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 759-769, 1992.

[66] A. Megretski, "Frequency-domain criteria of robust stability for slowly time-varying systems". *IEEE Transactions on Automatic Control*, vol. 40, no. 1, pp. 153-155, 1995.

[67] A. Megretski and A. Rantzer, "System analysis via intergral quadratic constraints". *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 819-830, 1997.

[68] P. J. Moylan, "Implications of passivity in a class of nonlinear systems". *IEEE Transactions on Automatic Control*, vol. 19, pp. 373-381, 1974.

[69] P. J. Moylan and D. J. Hill, "Tests for stability and instability of interconnected systems". *IEEE Transactions on Automatic Control*, vol. 24, no. 4, pp. 574-575, 1979.

[70] P. J. Moylan and D. J. Hill, "Stability criteria for large-scale systems". *IEEE Transactions on Automatic Control*, vol. 23, no. 2, pp. 143-149, 1978.

[71] A. Packard and J. Doyle, "Quadratic stability with real and complex perturbations". *IEEE Transactions on Automatic Control*, vol. 35, no. 2, pp. 198-201, 1990.

[72] G. Papageorgiou and K. Glover, "Design, development and control of the HIRM wind tunnel model". *Proceedings of the 38th IEEE Conference on Decision and Control*, vol. 2, pp. 1529-1537, Phoenix, Arizona, USA, 1999.

[73] L. S. Pontryagin, "Optimal regulation processes". *American Math Society Translations*, vol. 18, pp. 321-339, 1961.

[74] V. M. Popov, "Absolute stability of nonlinear systems of automatic control". *Automation and Remote Control*, vol. 22, pp. 857-875, 1962.

[75] L. Qiu and E. J. Davison, "Feedback stability under simultaneous gap metric uncertainties in plant and controller". *Systems & Control Letters*, vol. 18, pp. 9-22, 1992.

[76] R. Ravi, A. M. Pascoal and P. P. Khargonekar, "Normalized coprime factorizations for linear time-varying systems". *Systems & Control Letters*, vol. 18, no. 6, pp. 455-465, 1992.

[77] C. E. Rohrs, L. Valavani, M. Athans and G. Stein, "Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics". *IEEE Transactions on Automatic Control*, vol. 30, no. 9, pp. 881-889, 1985.

[78] I. W. Sandberg, "Some results on the theory of physical systems governed by nonlinear functional equations". *Bell System Technical Journal*, vol. 44, pp. 871-898, 1965.

[79] I. W. Sandberg, "On the $\mathcal{L}_2$-boundedness of solutions of nonlinear functional equations". *Bell System Technical Journal*, vol. 43, pp. 1581-1599, 1964.

[80] J. Shamma, "Robust stability with time-varying structured uncertainty". *IEEE Transactions on Automatic Control*, vol. 39, no. 4, pp. 714-724, 1994.

[81] F. Shimizu and S. Hara, "A method of structure/control design integration based on finite frequency conditions and its application to smart arm structure design". *Proceedings of the 41st SICE Annual Conference*, pp. 1123-1128, Osaka, Japan, 2002.

[82] A. R. Teel, T. T. Georgiou, L. Praly and E. Sontag, "Input-output stability" in: W. S. Levine (Ed.), *The Control Handbook*, pp. 895-908. CRC Press, Boca Raton, Florida, 1996.

[83] J. G. VanAntwerp, R. D. Braatz and N. V. Sahinidis, "Globally optimal robust process control". *Journal of Process Control*, vol. 9, pp. 375-383, 1999.

[84] M. Vidyasagar, *Nonlinear systems analysis.* Prentice Hall, Englewood Cliffs, New Jersey, 1993.

[85] M. Vidyasagar, "On the well-posedness of large-scale interconnected systems". *IEEE Transactions on Automatic Control*, vol. 25, no. 3, pp. 413-421, 1980.

[86] G. Vinnicombe, *Uncertainty and feedback: $\mathcal{H}_\infty$ loop-shaping and the ν-gap metric.* Imperial College Press, London, 2001.

[87] G. Vinnicombe, "A ν-gap distance for uncertain and nonlinear systems". *Proceedings of the 38th IEEE Conference on Decision and Control*, pp. 2557-2562, Phoenix, Arizona, USA, 1999.

[88] G. Vinnicombe, "On IQCs and the ν-gap metric". *Proceedings of the 37th IEEE Conference on Decision and Control*, pp. 1199-1200, Tampa, Florida, USA, 1998.

[89] G. Vinnicombe, "On closed-loop objectives and guaranteed robustness properties". *Proceedings of the 35th IEEE Conference on Decision and Control*, pp. 1869-1874, Kobe, Japan, 1996.

[90] G. Vinnicombe, "Frequency domain uncertainty and the graph topology". *IEEE Transactions on Automatic Control*, vol. 38, no. 9, pp. 1371-1383, 1993.

[91] J. C. Willems, "Mechanisms for the stability and instability in feedback systems". *Proceedings of the IEEE*, vol. 64, pp. 24-35, 1976.

[92] J. C. Willems, "Qualitative behavior of interconnected systems". *Annals of Systems Research*, vol. 3, pp. 61-80, 1973.

[93] J. C. Willems, "Dissipative dynamical systems, Part I: General theory; Part II: Linear systems with quadratic supply rates". *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321-393, 1972.

[94] J. C. Willems, "The generation of Lyapunov functions for input-output stable systems". *SIAM Journal on Control*, vol. 9, pp. 105-133, 1971.

[95] J. C. Willems, *The analysis of feedback systems.* The M.I.T. Press, Cambridge, Massachusetts, 1971.

[96] V. A. Yakubovich, "Absolute stability of nonlinear control in critical cases, Parts I and II". *Automation and Remote Control*, vol. 24, pp. 273-282, 655-688, 1963.

[97] V. A. Yakubovich, "The solution of certain matrix inequalities in automatic control theory". *Soviet Mathematics Doklady*, vol. 3, pp. 620-623, 1962.

[98] H. Yamauchi, S. Hara and T. Iwasaki, "Characterization of robust finite frequency positive-real property". *Proceedings of the American Control Conference*, pp. 2910-2711, Arlington, Virginia, USA, 2001.

[99] G. Zames, "Input-output feedback stability and robustness, 1959-85". *IEEE Control Systems Magazine*, vol. 13, no. 3, pp. 61-66, 1996.

[100] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems, Part I: Conditions derived using concepts of loop gain, conicity and positivity". *IEEE Transactions on Automatic Control*, vol. 11, pp. 228-238, 1966.

[101] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems, Part II: Conditions involving circles in the frequency plane and sector nonlinearities". *IEEE Transactions on Automatic Control*, vol. 11, pp. 465-476, 1966.

[102] G. Zames, "On the stability of nonlinear, time-varying feedback systems". *Proceedings of the National Electronics Conference*, vol. 20, pp. 725-730, 1964.

[103] G. Zames, "Functional analysis applied to nonlinear feedback systems". *IEEE Transactions on Circuits and Systems*, vol. 10, no. 3, pp. 392-404, 1963.

[104] K. Zhou with J. C. Doyle and K. Glover, *Robust and optimal control.* Prentice Hall, Upper Saddle River, New Jersey, 1996.

[105] S. Q. Zhu, M. L. J. Hautus and C. Praagman, "Sufficient conditions for robust BIBO stabilization: given by the gap metric". *Systems & Control Letters*, vol. 11, pp. 53-59, 1988.